

On the Optimal Precoding for MISO-WSN: One Time Slot Detection of Multiple Binary Data on the Same Frequency Band

Pouya Ghofrani and Anke Schmeink

Abstract—A fast multi-target detection technique—using a devised precoding in a wireless sensor network (WSN)—is proposed. The targets are detected by binary wireless sensors which only have one or zero logical outputs, indicating a target's presence or absence. The goal is to simultaneously decode all sensors' data in one time slot at the receiver node, where all sensors transmit data simultaneously on the same frequency band. For this to happen, a specific geometric deployment of the sensors is presented in the paper. The concept in this work might have similarities to the bijective mappings in the superposition modulation. Different channel gains and the additive white Gaussian noise are included in the model. The system model is described in detail and the communication scheme to fulfill a simultaneous decoding is mathematically formulated. Then, the parametric solutions to this general digital modulation problem are studied step by step through several theorems, and the optimal solutions that minimize the detection error are found. The combined effects of the noise and the non-perfect channel state information on the signal constellations are both analytically and numerically investigated. In addition, two other detection scenarios with a priori information about the targets are studied.

Index Terms—Bijective mapping, signal constellation, superposition modulation, wireless sensor networks (WSNs).

I. INTRODUCTION

Sensor networks for surveillance and detection applications have been investigated in many research works, covering a plethora of system designs with a variety of objectives and constraints for each specific application. In spite of this diversity, there are some scenarios where the binary data of all sensors should be decoded at the same time in order to have instant control over the phenomenon. In other words, the very little delay induced by time division multiplexing (TDM) schemes might have an irreversible impact on the detection or tracking performance. Examples of such scenarios include: studying the behavior of particles or substances undergoing experiments such as extreme electromagnetic excitations, live tracking and controlling the intensively accelerated particles emanated from instant chemical reactions, accurate shockwave pattern acquisition, and periphery protection. Although this might be achieved by using a conventional frequency division multiplexing (FDM) scheme, it might not be practical in large

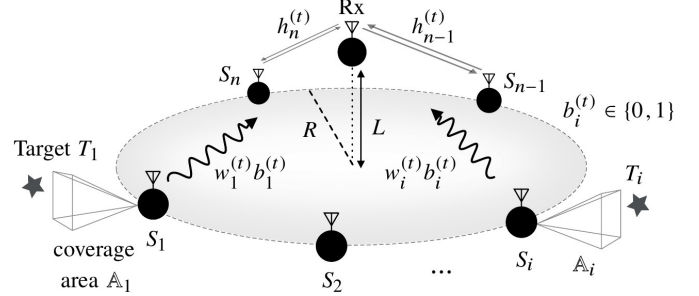


Fig. 1. A circular deployment of binary wireless sensors and the sink node (Rx) somewhere on the 3D locus of points equidistant from the binary sensors to ensure a dominant line-of-sight propagation for all sensors (figure modified from [1]).

networks consisting of many nodes. The main reason for this is that in large networks, the unique frequency allocation to each sensor might not be possible. This is generally due to the hardware limitations or the interference with other sources. Furthermore, the sensor replacement in this case is not simple, as each sensor operates on a unique frequency band.

In this work, we investigate an optimal precoding scheme to resolve the above-mentioned issues with the TDM and FDM schemes for fast detection applications. For the sake of compactness and to avoid ambiguity, in this paper, we define $\mathbb{F}_i^j := \{i, i+1, \dots, j\}$ for every $i, j \in \mathbb{Z}$, where $i \leq j$. We consider a binary WSN (BWSN), as in Fig. 1, that is characterized as follows: all n binary sensors S_i , $i \in \mathbb{F}_1^n$, deployed for detection of up to n targets T_i , $i \in \mathbb{F}_1^n$, transmit data simultaneously on the same frequency band, the binary data of each sensor is determined depending on the targets presence or absence, the independent targets each have an equal presence-absence probability, and each sensor S_i covers a known area, e.g., A_i , for the detection of target T_i . Note that introducing A_i and T_i here is only to match to a practical implementation. In other words, the mathematical model in this paper takes into account the final data acquired by the sensors in each instant, regardless of how the data was gathered from a target T_i which appeared in A_i .

A. The Conceptual Connections with Other Techniques

There are similarities between the proposed method and the conventional communication techniques such as code division multiple access (CDMA) (see e.g. [2], [3]), distributed beamforming (DBF) and superposition modulation (SuM). The similarities mainly fall in the scope of the multiplexing concept. In

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P. Ghofrani and A. Schmeink are with the ISEK teaching and research area, RWTH Aachen University, Aachen, Germany (e-mail: {pouya.ghofrani, anke.schmeink}@isek.rwth-aachen.de).

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a CDMA system, several users share the same communication channel and frequency band, where they can simultaneously transmit data using different codes. The receiver will then extract each user's data from the received signal using these codes. The major difference between the proposed scheme and CDMA is as follows. The CDMA is generally a spread-spectrum method, where for any user at the receiver, other users' data appear like noisy signals which do not correlate with that user's code. Contrary to this, in the proposed model, all sensors operate on the same narrow frequency band, and the interference with other sensors' data is constructive.

Another similar yet different technique is DBF. The conventional beamforming (BF) is a technique in which several antennas in a specifically arranged array focus the beam to the receiver's direction. This is mainly to increase the signal-to-noise ratio (SNR) at the receiver and suppress the interference with other users. Thus, in the BF technique, the antennas at the transmitter need to cooperate in amplitude and phase to shape the beam. It is clear that in this work, there is no cooperation between the sensors as they are distributed over an area without any cross communication link. Therefore, a comparison with distributed beamforming might be more relevant in this context. However, DBF can convey different meanings in different scenarios. In the literature of wireless sensor networking, distributed beamformers are usually modeled as wireless relay channels [4]. In [4], a review of DBF is presented, and in [5], a suboptimal solution for DBF and power allocation is proposed, where the cooperative communication system consists of one source node, one destination node, and several relays. Although with these descriptions our proposed scheme in this work has some similarities to the DBF technique due to devising the code vector, the main difference is in their objectives. In fact, we aim at minimizing the time, whereas in DBF, mainly the SNR maximization is intended. In addition, contrary to the suboptimal solutions in DBF, our solution here is optimal, since the whole data vector is decoded within only one time step. Note that this is the least time possible.

Considering the general description of SuM in [6], there are fundamental differences between the proposed scheme and the SuM. First, the SuM uses binary phase shift keying (BPSK) modulation, whereas the proposed precoding here uses a generalized version of on-off keying (OOK) at the transmitters. Second, the implementation of the SuM requires a sum module in order to perform the linear superposition of the weighted symbols, whereas in this work, there is no physical sum module in the system, as the final symbols are generated by wave superposition at the receiver. Third, the SuM operates on parallel data streams, which in fact come from one main binary stream, whereas here, the sensors' data are from independent sources. Nevertheless, the proposed scheme and the SuM might overlap when an exponential weighting is used in the SuM. This is known as a special case of unequal power allocation (UPA) in the related literature. In spite of this mild similarity, and in addition to the points above, there is a noticeable difference between the objective of the SuM and the aim of our scheme in this work. Indeed, similar to other conventional mappings, the SuM is devised and evaluated

mainly based on the capacity achieving and coding criteria, whereas in this work, we aim at minimizing the transmission and detection time. Above all, the similarities discussed so far can only make sense when the SuM is bijective, as our proposed scheme is based on bijective mappings. Note that a non-bijective mapping in this work causes ambiguity in target detection and therefore, should be avoided.

Unlike the previously discussed techniques, one definition of network coding might overlap with the proposed model in this paper from the objective point of view. More precisely, a network coding employed for increasing network throughput by communicating more information while using fewer packet transmissions [7] can have similarities to the work presented here from the delay reduction perspective. However, a network coding which is defined as the coding applied to the contents of packets in a packet network is a broad subject that is conducted above the physical layer [7], whereas the proposed scheme in this work lies entirely in the physical layer. In addition, the coding in network coding is often devised by algebraic algorithms in order to reduce the number of transmissions in a network. The data flowing in such networks is often conducted via multiple relay nodes that apply the coding on the data. Contrary to this, our model here does not involve any relaying via other nodes, and there is a direct source-destination link for all nodes. In terms of literature and mathematical areas such as real and discrete domains, the differences between the proposed scheme and the network coding approaches can be clearly seen in the relevant works such as [8] and [9].

So far, we have compared the proposed scheme, that involved an OOK mapping, with other similar techniques such as CDMA, DBF, SuM and network coding, separately. However, several research works have considered different combinations of these techniques for specific applications. In [10], a coded modulation scheme is proposed based on block Markov superposition transmission, where an OOK mapping and a SuM model are employed to form an equispaced pulse amplitude modulation constellation for an indoor visible light communication scenario. A superposition constellation design for wireless network coding is proposed in [11], and a signature design of sparsely spread CDMA (SCDMA) based on superposed constellation distance analysis is proposed in [12]. The differences between our approach and the techniques above can be found based on the previous explanations.

As we will see later, due to the bijective property of the precoding investigated in this paper, it shares common concepts with sum-distinct sets studied in [13]–[18]. However, as we will see in the next sections, the optimal precoding for our system requires only those solutions of sum-distinct sets that minimize the detection error at the receiver in the presence of noise and the channel estimation errors. Therefore, the problem formulation here and our approach toward the solution is different from the works above.

B. Major Contributions

We discuss a model for *specific* delay-critical applications, where using a shared narrow-band channel for all transmitting sensors, *the one time slot detection of the whole sensors'*

data is the only objective. In other words, *we can spend as much power as required* in order to achieve this goal, though we typically try to keep it as low as possible. In the next step, among all the solutions—which will later be called the *amplitude allocations*—that allow a one time slot detection, we aim at finding those which minimize the detection error. Besides the fast detection goal, this minimum error property is the main difference with other relevant works, such as [13]–[18]. The system model description is as close as possible to the practical scenarios (mainly line-of-sight) as it contains both the channel model and the additive white Gaussian noise (AWGN). The joint effect of the noise and the non-perfect channel state information (CSI) on the constellations is analytically investigated using highly accurate approximations that are followed by computer simulations. The theorems and the proofs in this work make the major contribution to the usage of this model. To broaden the range of applications, we then consider other scenarios in which the number of targets is less than n . Note that we mainly consider the model described in Fig. 1 only for the simplicity of discussing the concepts by visualization. In fact, the proposed scheme can be used in other optical and wired systems as well, as long as the pulse width, distances, and the angular design of the sensor deployment are determined as explained in [19]. A summary of the major contributions is as follows

- proposing a detection scenario with the discussed characteristics,
- mathematical formulation of the model considering the noise and channel coefficients,
- deriving the bijective mappings or analyzing their existence in order to simultaneously detect the data of the sensors with minimum detection error,
- analytical and numerical investigation of the effects of the channel errors and noise on the signal constellations, and
- simulation of the detection accuracy in a general case.

The rest of this paper is organized as follows: In Section II, the BWSN system model is described first and the basic parametric precoding scheme for the problem is derived based on our previous work in [1]. Then, this parametric model is thoroughly investigated in Section II-A in order to obtain the optimal solution for our BWSN model. In Section III, the effect of the non-perfect CSI is studied in detail with analytical derivations and numerical simulations. The possibility of an extension to other interesting target detection scenarios is discussed in Section IV and finally, a performance evaluation is conducted by numerical simulations in Section V, where the combined effects of the noise and the non-perfect CSI on the performance are investigated.

II. THE PROPOSED BWSN SYSTEM MODEL

Similar to a multiple input single output (MISO) scenario, the received signal at the sink node, denoted by r_t , can be expressed as $r_t = \mathbf{h}_t^T (\mathbf{w}_t \circledast \mathbf{b}_t) + n_t$, where $\mathbf{h}_t := [h_1^{(t)}, h_2^{(t)}, \dots, h_n^{(t)}]^T \in \mathbb{R}^n$ is the vector of channel coefficients, $\mathbf{w}_t := [w_1^{(t)}, w_2^{(t)}, \dots, w_n^{(t)}]^T \in \mathbb{R}^n$ is the weighting vector we aim to devise, symbol \circledast shows the Hadamard

product, $\mathbf{b}_t := [b_1^{(t)}, b_2^{(t)}, \dots, b_n^{(t)}]^T \in \{0, 1\}^n$ is the vector containing the binary data, and $n_t \in \mathbb{R}$ denotes the AWGN with the distribution $\mathcal{N}(0, \sigma_n^2)$. In the definitions above, letter $t \in \mathbb{N}$ in the subscripts of vectors and superscripts of entries denotes an integer indicating the time-slot index. In order to devise the weighting vector \mathbf{w}_t , we need to estimate the channel vector \mathbf{h}_t . A fast converging channel estimation algorithm for WSNs is proposed in [20]. However, according to the thorough investigation in our previous work [1], where different solutions are discussed, we consider a reciprocal channel response for downlink and uplink over a very short period of time. Hereafter, we refer to this estimated channel as vector $\hat{\mathbf{h}}_t := [\hat{h}_1^{(t)}, \hat{h}_2^{(t)}, \dots, \hat{h}_n^{(t)}]^T \in \mathbb{R}^n$ that is entry-wise available at each corresponding transmitter node. We also denote the Hadamard division by \oslash , and the reminder when a is divided by b by $\text{mod}(a, b)$. Defining $\mathbf{v}_t := \beta \mathbf{v}_t \oslash \hat{\mathbf{h}}_t$, where $\mathbf{v}_t := [\rho^{\text{mod}(t, n)}, \rho^{\text{mod}(1+t, n)}, \dots, \rho^{\text{mod}(n-1+t, n)}]^T$ is the *proposed amplitude allocation vector* with $\rho \in \mathbb{F}_2^\infty$ for now, and $0 < \beta \leq 1$ is a known parameter to all sensors, the received signal can be expressed as follows

$$r_t = \mathbf{h}_t^T (\beta \mathbf{v}_t \oslash \hat{\mathbf{h}}_t \circledast \mathbf{b}_t) + n_t \quad (1)$$

$$= \beta (\mathbf{h}_t^T \oslash \hat{\mathbf{h}}_t^T) (\mathbf{v}_t \circledast \mathbf{b}_t) + n_t \quad (2)$$

$$\approx \beta [1, \dots, 1] (\mathbf{v}_t \circledast \mathbf{b}_t) + n_t \quad (3)$$

$$= \beta \sum_{i=1}^n \rho^{\text{mod}(i-1+t, n)} b_i^{(t)} + n_t \quad (4)$$

$$= \beta (\overline{b_{k_{n-1}}^{(t)} b_{k_{n-2}}^{(t)} \dots b_{k_0}^{(t)}})_\rho + n_t, \quad (5)$$

where for every $j \in \mathbb{F}_0^{n-1}$, $k_j := \arg [\text{mod}(i-1+t, n) = j]$, $i \in \mathbb{F}_1^n$

and $\overline{b_{k_{n-1}}^{(t)} b_{k_{n-2}}^{(t)} \dots b_{k_0}^{(t)}}_\rho$ in Eq. (5) represents the decimal $\sum_{i=1}^n \rho^{\text{mod}(i-1+t, n)} b_i^{(t)}$ in a base- ρ system. Therefore, Eq. (5) shows that the whole data vector \mathbf{b}_t , that contains the data of all sensors, can be detected using only the information of the received scalar r_t . Note that the accuracy of the approximation in Eq. (3) just depends on the quality of the CSI and channel equalization. In the relations above, β was used to avoid transmitting high amplitudes after the Hadamard division by $\hat{\mathbf{h}}_t$. This is analogous to the problems of zero-forcing channel equalization. In fact, this scaling parameter makes the implementation comply with the practical requirements which are often due to the hardware limitations. Since the channel vector in our model includes the combined effects of all attenuation parameters and in particular, the power decay with distance, the value of $\sqrt{R^2 + L^2}$ with R and L defined in Fig. 1 plays an important role in setting β , as $|h_i^{(t)}|^2$ ($i \in \mathbb{F}_1^n$) for the line-of-sight (LOS) components is proportional to $1/(\sqrt{R^2 + L^2})^{\gamma_{PL}}$, where γ_{PL} denotes the path-loss exponent. Note that when the carrier signal is in the radio frequency range, due to the relatively large-scale constructive and destructive interferences, the geometry of the network in Fig. 1, i.e., the circular deployment of the sensors, is of high importance. Here, we note that the periodicity $\mathbf{v}_{t+mn} = \mathbf{v}_t$ for every $m \in \mathbb{Z}_{\geq 0}$ comes from the

circular shift property of the entries at different instants. This plays an important role in uniformly distributing the energy consumption among the sensors. The relations above exhibit that from a theoretical point of view, there is no constraint on the number of sensors (n) for this instant fusion. However, the exponential growth of the amplitude allocation vector with respect to n restricts this instant fusion scheme for a large number of sensors in highly noisy environments. The reason for this is that under noisy conditions, a relatively small β cannot be effective, since it amplifies the noise for decoding at the receiver. Therefore, in such cases, the main load of pulse shaping to form the weighting vector \mathbf{w}_t should be carried by the power amplifiers, while their output dynamic range is limited and cannot support arbitrarily large amplitudes.

Note that in this work, the binary data vector \mathbf{b}_t was directly used in the coding process without any pre-mapping onto other symbols such as binary antipodal symbols. The reason behind this idea is to exploit the energy conservation advantage of the OOK. This means that the sensors with no detected targets at time t do not transmit any signal. In the following, the *amplitude allocation set* is defined as a set consisting of all entries of the corresponding amplitude allocation vector. In other words, if $\mathbf{w}_t := \beta(\mathbf{x}_t \odot \mathbf{h}_t^T)$, where $\mathbf{x}_t := [x_1^{(t)}, \dots, x_n^{(t)}]^T$, the set $\mathbb{X} := \{x_i^{(t)} | i \in \mathbb{F}_1^n\}$ is an amplitude allocation set. Note that contrary to \mathbf{x}_t , set \mathbb{X} is independent of time. The reason is that in this work, we use a cyclic shift of $x_i^{(t)}$ s for the next time steps and therefore, devising \mathbf{x}_t and \mathbb{X} are equivalent. Thus, in the following discussions, due to the simplicity of time invariance, devising set \mathbb{X} is considered.

A. Choosing the Value of ρ

So far, we introduced parameter ρ as an element of \mathbb{F}_2^∞ . However, in order to be able to uniquely reconstruct the data vector \mathbf{b}_t (or $\mathbf{v}_t \otimes \mathbf{b}_t$) from the scalar r_t , i.e., to have a bijective mapping (BM) between the data vector and the received scalar, ρ can take many values.

Theorem 1. *For a BWSN with n binary sensors, and regardless of the mapping scheme at the transmitter (so far, we used OOK, as \mathbf{b}_t consists of zeros and ones), there are infinite choices for $\rho \in \mathbb{R}$ to construct a BM between the transmitter and the receiver. For example, any sum combinations of $\{1, \rho, \dots, \rho^{n-1}\}$ for $\rho = 0.3$ creates a distinct scalar number, and therefore, it creates a BM between the transmitter and the receiver.*

Proof. See Appendix A.

Mathematically speaking, a general solution to the problem is provided by the following theorem.

Theorem 2. *For any $\rho \in \mathbb{Q} \setminus \{0, \pm 1\}$ and $n \in \mathbb{N}$, the amplitude allocation set $\{1, \rho, \dots, \rho^{n-1}\}$ for n binary sensors creates a BM.*

Proof. See Appendix B.

Considering the fact that there exists a rational number in any neighborhood of a real number, the importance of Theorem 2 is more highlighted as any hardware dictated real-valued choice for ρ can be approximated with arbitrary precision while the bijective property is preserved. However, for most

of the possible ρ values which form a BM, the constellation points are unequally spaced. In our MISO scenario, a combination of different OOK schemes, i.e., with different transmit amplitudes, are superimposed by the electromagnetic waves at the receiver, where an amplitude shift keying (ASK) modulation with 2^n constellation points (2^n -ASK) is formed. Based on the detailed explanation about the accuracy of the channel estimation model in [1], the constellation points should be equispaced. Since the digital modulation scheme in this work is based on amplitude modulation, we use the terms ASK and BM interchangeably. Denoting a BM with n_c constellation points by n_c -BM, and similarly, an equispaced BM (EBM) with n_c constellation points by n_c -EBM, the equally spaced property to have an optimal signal constellation is proved in the following theorem. Note that the proof in Appendix C will be reused further in Section III.

Theorem 3. *If $b_1^{(t)}, b_2^{(t)}, \dots, b_n^{(t)}$ are mutually independent and identically distributed (i.i.d.) discrete random variables with the uniform distribution $\mathcal{U}\{0, 1\}$, the detection error probability is minimized for a 2^n -EBM.*

Proof. See Appendix C.

Considering the theorem above, our objective is to find all possible solutions. In other words, analogous to the optimal codebook designs, e.g., in [21], and the optimal matrix constellations [22], we aim at finding the amplitude allocations which result in the corresponding EBMs. The following theorem states a necessary and sufficient condition to obtain them.

Theorem 4. *For a BWSN with n binary sensors, a 2^n -EBM at the receiver exists, if and only if the employed amplitude allocation set is in the form of $\{\eta t_i 2^{i-1} | i \in \mathbb{F}_1^n\}$, where $\eta > 0$ and $t_i = \pm 1$, for every $i \in \mathbb{F}_1^n$.*

Proof. See Appendix D.

III. THE EFFECT OF THE NON-PERFECT CSI

So far, we have assumed that a close to perfect CSI is available, which is a valid assumption based on the explanations in the previous sections. In this section, we investigate how a non-perfect CSI affects the performance in terms of constellation change. For clearness, we define the i^{th} digit of integer A in base $b \in \mathbb{Z}_{\geq 2}$ as $\kappa_{(A)_b}^{(i)} := a_i$, where $A = (\overline{a_n a_{n-1} \dots a_1})_b$ and $a_i \in \mathbb{F}_b^{b-1} \forall i \in \mathbb{F}_1^n$. Considering the amplitude allocation set $\{1, \rho, \dots, \rho^{n-1}\}$ and Fig. 6 in Appendix C, the value of constellation point i , denoted by m_i , can be expressed as follows

$$m_i = \beta \sum_{j=1}^n \kappa_{(i-1)_2}^{(j)} \rho^{j-1} \quad \forall i \in \mathbb{F}_1^{2^n}. \quad (6)$$

Note that from Eq. (6), the values of α_i in Fig. 6 are already given. However, because of the non-perfect CSI, each m_i has a different error term which is induced by the channel estimation errors at the sensor nodes. Therefore, these different error distributions then redefine the problem to find the optimum α_i s. Thus, for each constellation point m_i expressed in Eq. (6), we define ξ_i as the sink node-transferred error which is caused by the channel estimation errors (e_i s) associated with m_i . In other words, for every $i \in \mathbb{F}_1^{2^n}$,

$$\begin{aligned} \xi_i &:= \beta(\mathbf{h}_t^T \odot \hat{\mathbf{h}}_t^T - [1, 1, \dots, 1])(\mathbf{v}_t \otimes \mathbf{b}_t) \\ &\text{when } \beta[1, 1, \dots, 1](\mathbf{v}_t \otimes \mathbf{b}_t) = m_i. \end{aligned} \quad (7)$$

Considering a constant channel $\mathbf{h}_t := [h, h, \dots, h]^T$ for simplicity, the relations below follow

$$\xi_i = \beta(\mathbf{h}_t^T \odot \hat{\mathbf{h}}_t^T - [1, 1, \dots, 1])(\mathbf{v}_t \otimes \mathbf{b}_t) \quad (8)$$

$$= \beta([h, \dots, h] \odot [h + e_1, \dots, h + e_n] - [1, 1, \dots, 1])(\mathbf{v}_t \otimes \mathbf{b}_t) \quad (9)$$

$$= \beta\left[\frac{1}{1 + e_1/h} - 1, \dots, \frac{1}{1 + e_n/h} - 1\right](\mathbf{v}_t \otimes \mathbf{b}_t) \quad (10)$$

$$\approx \beta[-e_1/h, \dots, -e_n/h](\mathbf{v}_t \otimes \mathbf{b}_t) \quad (11)$$

$$= -\beta \sum_{j=1}^n \kappa_{(i-1)_2}^{(j)} \frac{e_{q_j}}{h} \rho^{j-1}, \text{ where } \{q_1, \dots, q_n\} = \mathbb{F}_1^n. \quad (12)$$

In the derivations above, Eq. (11) is obtained from Eq. (10) using the Maclaurin series approximation $\frac{1}{1+x} \approx 1 - x$ for small x values, and Eq. (12) is deduced using the relation $\beta[1, 1, \dots, 1](\mathbf{v}_t \otimes \mathbf{b}_t) = \beta \sum_{j=1}^n \kappa_{(i-1)_2}^{(j)} \rho^{j-1}$ based on Eqs. (6,7). Note that in Eq. (12), elaborating on the exact value of each q_j is not necessary, as e_i s are Gaussian i.i.d. random variables. Assuming a constant variance $\sigma_{e_i}^2 = \sigma_e^2$ for every $i \in \mathbb{F}_1^n$, where $e_i \sim \mathcal{N}(0, \sigma_e^2)$, we obtain

$$-\beta \sum_{j=1}^n \kappa_{(i-1)_2}^{(j)} \frac{e_{q_j}}{h} \rho^{j-1} \sim \mathcal{N}\left(0, \beta^2 \sum_{j=1}^n \left(\kappa_{(i-1)_2}^{(j)} \frac{\sigma_e}{h} \rho^{j-1}\right)^2\right)$$

$$\stackrel{\text{Eq. (12)}}{\implies} \sigma_{\xi_i}^2 \approx \frac{\beta^2 \sigma_e^2}{h^2} \sum_{j=1}^n \left(\kappa_{(i-1)_2}^{(j)}\right)^2 \rho^{2(j-1)} \quad (13)$$

$$= \frac{\beta^2 \sigma_e^2}{h^2} \sum_{j=1}^n \kappa_{(i-1)_2}^{(j)} \rho^{2(j-1)}. \quad (14)$$

For convenience, the main notations are summarized in Tab. I. Denoting the conditional probability by $Pr(\cdot)$, and defining

$$\forall i \in \mathbb{F}_1^{2^n} : f_i(\lambda) := (2\pi(\sigma_{\xi_i}^2 + \sigma_e^2))^{-0.5} \exp\left(\frac{-\lambda^2}{2(\sigma_{\xi_i}^2 + \sigma_e^2)}\right),$$

the probability of erroneous detection, Pr_E , can be expressed as $Pr_E = 1 - Pr_S$, where the probability of successful detection, Pr_S , can be written as

$$Pr_S = \sum_{i=1}^{2^n} Pr(m_i + \xi_i + \epsilon \in \mathbb{M}_i | m_i) Pr(m_i) \quad (15)$$

$$= \sum_{i=1}^{2^n} Pr(\epsilon + \xi_i \in \mathbb{M}_i - m_i) \prod_{j=1}^n \frac{1}{2} \quad (16)$$

$$= \frac{1}{2^n} \left(\sum_{i=1}^{2^n-2} \int_{-\alpha_i}^{\alpha_{i+1}} f_{i+1}(\lambda) d\lambda + \int_{-\infty}^{\alpha_1} f_1(\lambda) d\lambda + \int_{-\alpha_{2^n-1}}^{\infty} f_{2^n}(\lambda) d\lambda \right).$$

In Eq. (15), the terms of the form $Pr(\cdot | m_i) Pr(m_i)$, or equivalently $Pr(\cdot \cap m_i)$, are the joint probability of occurrence of “.” and transmitting amplitudes which make m_i at the receiver. Similar to Eqs. (28,29,30) in Appendix C, the optimization problem here can be formulated as

$$\begin{aligned} & \text{maximize:} && G(\alpha_1, \dots, \alpha_{2^n-2}) \\ & \text{subject to:} && \sum_{i=1}^{2^n-2} \alpha_i \leq \frac{D}{2}, \forall i \in \mathbb{F}_1^{2^n-2} \alpha_i \geq 0, \end{aligned} \quad (17)$$

TABLE I
NOTATION SUMMARY.

Notation	Description
\mathbb{F}_i^j	set of integers $\{i, i+1, \dots, j\}$;
$x, \mathbf{x}, \mathbf{X}$	scalar x , vector \mathbf{x} , matrix \mathbf{X} ;
\otimes, \odot	Hadamard product, Hadamard division;
$\text{mod}(a, b)$	the remainder when a is divided by b ;
$(a_n, a_{n-1}, \dots, a_1)_b$	decimal $\sum_{i=1}^n a_i b^{i-1}$ in a base- b system;
$m_i (i \in \mathbb{F}_1^{2^n})$	the value of the i^{th} constellation point;
$e_i (i \in \mathbb{F}_1^n)$	error of channel estimation at sensor i ;
$\xi_i (i \in \mathbb{F}_1^{2^n})$	channel estimation errors affecting m_i ;
ϵ	AWGN component of the sink node;
D	maximum Euclidean distance (Figure 6);
$\sigma_e^2, \sigma_{e_i}^2, \sigma_{\xi_i}^2$	variances of ϵ, e_i and ξ_i , respectively;
$\alpha_i (i \in \mathbb{F}_1^{2^n-1})$	positive variables (as shown in Figure 6);
$\mathbb{M}_i (i \in \mathbb{F}_2^{2^n-1})$	$(m_i - \alpha_{i-1}, m_i + \alpha_i]$ (detect. of m_i);
\mathbb{M}_1	$(-\infty, m_1 + \alpha_1]$ (detect. of m_1);
\mathbb{M}_{2^n}	$(m_{2^n} - \alpha_{2^n-1}, +\infty)$ (detect. of m_{2^n});
$\mathbb{S}(\mathbb{X}), \mathbb{S}_{n_T}(\mathbb{X})$	functions defined in Eqs. (35,62), resp.;
$\mathbb{T}(\mathbb{X}), \mathbb{D}(\mathbb{X})$	functions defined in Eqs. (37,50), resp.;
$\mathbb{S}(\mathbb{X}), \mathbb{T}(\mathbb{X}), \mathbb{D}(\mathbb{X})$ $\leq n_T \quad \leq n_T \quad \leq n_T$	functions defined in Eqs. (89,90,91), resp.

where function $G(\alpha_1, \dots, \alpha_{2^n-2})$ is defined as

$$G := \sum_{i=1}^{2^n-2} \int_0^{\alpha_i} (f_i + f_{i+1})_{(\lambda)} d\lambda + \int_0^{\frac{D}{2} - \sum_{j=1}^{2^n-2} \alpha_j} (f_{2^n-1} + f_{2^n})_{(\lambda)} d\lambda \quad (18)$$

and D in this case is $D = m_{2^n} - m_1 = \beta \sum_{i=1}^n \rho^{i-1} = \beta \frac{\rho^n - 1}{\rho - 1}$. The Hessian matrix $\nabla^2 G$ can be expressed as Eq. (19). Similar to Eqs. (33,34), the two matrices in Eq. (19) are each negative definite and therefore $\nabla^2 G \prec 0$, i.e., G is maximized when $\nabla G = \mathbf{0}$. Thus, for every $j \in \mathbb{F}_1^{2^n-2}$, it holds that

$$\begin{aligned} \frac{\partial G}{\partial \alpha_j} &= (f_{j+1} + f_j)_{(\alpha_j)} - (f_{2^n-1} + f_{2^n})_{(\frac{D}{2} - \sum_{i=1}^{2^n-2} \alpha_i)} = 0 \\ &\Rightarrow (f_{j+1} + f_j)_{(\alpha_j)} = (f_{2^n-1} + f_{2^n})_{(\frac{D}{2} - \sum_{i=1}^{2^n-2} \alpha_i)}. \end{aligned} \quad (20)$$

Analytically solving the set of equations in Eq. (20) is difficult. However, in order to illustrate the effect of the non-perfect CSI, we solve these equations numerically with parameters $n = 3, \beta = 1, \rho = 2, \sigma_e^2 = 0.01$ and $h = 0.5$. As shown in Fig. 2, the values of α_i change significantly as the variance of the channel estimation error (σ_e^2) grows, such that the distances between the constellation points increase in favor of the higher constellation values. This figure also shows that for any σ_e^2 , the order $\alpha_1 < \alpha_2 < \dots < \alpha_7$ is preserved. Figure 3 visually demonstrates the gradual transformation of the primary 8-EBM as σ_e^2 grows from zero to 5E-4. For convenience of comparison, the equispaced constellation, i.e., 8-EBM, is shown besides the optimum constellation in each step. We notice that such a behavior is quite expected, as the channel estimation errors are more amplified for constellation points with higher values. Note that the purpose of this

section is just to investigate the behavior of the proposed precoding when the channel estimation errors become non-negligible while ρ is fixed. In other words, for simplicity reasons, the joint effect of ρ and σ_e^2 — in Eq. (14)—on the optimal constellation is not studied here.

IV. OTHER DETECTION SCENARIOS

So far, we have studied a scenario in which 2^n constellation points were needed to uniquely decode each sensor's data, regardless of the number of targets. However, there are applications in which the number of present targets is always constant. In other words, in each instant, there are always n_T targets present at n_T sensors, and our purpose is to find these sensors. As a result, more efficient constellations should be devised, as only $\binom{n}{n_T}$ out of the total 2^n points are needed.

Theorem 5. *For a scenario, where in each instant, exactly n_T sensors out of the total n binary sensors ($n_T < n$) transmit data as they have detected targets, the existence of an $\binom{n}{n_T}$ -EBM is not guaranteed for arbitrary n_T and n values.*

Proof. See Appendix E.

Corollaries E.1 and E.2 in Appendix E are two important results of Theorem 5, which state that an $\binom{n}{n_T}$ -EBM does not exist for $(n_T, n) = \left(\overline{(t_S t_{S-1} \dots t_1 0)_2}, \overline{(v_S v_{S-1} \dots v_1 1)_2} \right)$, where $v_i, t_i \in \{0, 1\}$ and $S \in \mathbb{N}$ are such that $n_T < n$ and $\exists m \in \mathbb{F}_1^S \ni (t_m, v_m) = (1, 0)$, whereas an $\binom{n}{n_T}$ -EBM always exists for $n_T = 1$ or $n - 1$, regardless of the value of n .

Another potential scenario is when we have a priori knowledge about the maximum number of targets. So in this case, we attempt to find an $\sum_{i=0}^{n_T} \binom{n}{i}$ -EBM. This is addressed in Theorem 6, where an EBM on the ray $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$ means that all constellation points are non-negative, or non-positive, respectively.

Theorem 6. *For a scenario, where in each instant, at most n_T sensors out of the total n binary sensors ($n_T < n$) transmit data as they have detected targets (i.e., when the total number of targets does not exceed n_T), there exists no $\sum_{i=0}^{n_T} \binom{n}{i}$ -EBM on the ray $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$ when $2 \leq n_T \leq n - 2$.*

Proof. See Appendix F.

A result of Theorem 6 is that an $\sum_{i=0}^{n_T} \binom{n}{i}$ -EBM for the case $2 \leq n_T \leq n - 2$ may only exist when the amplitude allocation vector has both positive and negative entries. It is crucial to mention that the importance of the theorems on the existence

of EBMs, such as Theorems 5 and 6, is based on Theorem 3, which implicitly states that *if* an n_c -EBM *exists*, i.e., can be created by an amplitude allocation for a problem, then it is the optimum constellation among all n_c -EBMs in terms of detection-error minimization (note that Eq. (3) also holds for any arbitrary number of constellation points). In other words, if a theorem states that there is no EBM for some n_T and n values in a described scenario, it does not imply that the optimum constellation does not exist.

V. SIMULATIONS

In this section, we evaluate the performance of the proposed scheme in terms of detection accuracy. We consider a BWSN with $n = 3$ binary sensors for detecting three independent targets each with a uniform presence-absence probability. Although according to our previous work [1], the channel estimation error (CEE) is negligible, the simulations in this section are conducted, taking into account the effects of the CEEs for each binary sensor, as well as the AWGN of the sink node. The main reason for this comprehensive evaluation is the CEE amplifications occurred by \mathbf{v}_t . Therefore, since we include the effect of the CEE for each sensor independently, it is reasonable to assume that the actual channel vector $\mathbf{h}_t = [h_1^{(t)}, h_2^{(t)}, \dots, h_n^{(t)}]^T \in \mathbb{R}^n$ is in fact a constant vector, i.e., $\mathbf{h}_t = [h, h, \dots, h]^T \in \mathbb{R}^n$, in order to simplify the process. Moreover, parameter β is set to one here, as it is just a scaling factor that only changes the SNR. Thus, the vector of the normalized CEE (NCEE) which is defined as $NCEE := \mathbf{e}_t \oslash \mathbf{h}_t$ where $\mathbf{e}_t := \hat{\mathbf{h}}_t - \mathbf{h}_t$, can be expressed as $NCEE = \frac{1}{h} \mathbf{e}_t$. In order to show the tolerable region for error free detection, a simulation is conducted in which \mathbf{e}_t is fixed in the value of each entry, i.e., $\mathbf{e}_t = [\alpha, \alpha, \dots, \alpha]^T$, and thus, NCEE becomes $NCEE = [\frac{\alpha}{h}, \frac{\alpha}{h}, \dots, \frac{\alpha}{h}]^T$. Also for the sake of disambiguation, the fixed noise value (n_t) is denoted by θ . Figure 4 shows the detection performance for different values of $\frac{\alpha}{h}$ and θ , where a detailed contour plot of the figure for $\theta \in [-0.8, 0.8]$ is illustrated in Eq. (5). According to this figure, and considering the fact that the minimum Euclidean distance between the constellation points is one, there is a wide and almost-symmetric region of α/h and θ in which the target detection is error free.

$$\nabla^2 G = \begin{bmatrix} f'_1(\alpha_1) + f'_{2^n-1}(\alpha_{2^n-1}) & f'_{2^n-1}(\alpha_{2^n-1}) & \dots & f'_{2^n-1}(\alpha_{2^n-1}) \\ f'_{2^n-1}(\alpha_{2^n-1}) & f'_2(\alpha_2) + f'_{2^n-1}(\alpha_{2^n-1}) & \ddots & \vdots \\ \vdots & f'_{2^n-1}(\alpha_{2^n-1}) & \ddots & f'_{2^n-1}(\alpha_{2^n-1}) \\ f'_{2^n-1}(\alpha_{2^n-1}) & \dots & f'_{2^n-1}(\alpha_{2^n-1}) & f'_{2^n-2}(\alpha_{2^n-2}) + f'_{2^n-1}(\alpha_{2^n-1}) \end{bmatrix} + \begin{bmatrix} f'_2(\alpha_1) + f'_{2^n}(\alpha_{2^n-1}) & f'_{2^n}(\alpha_{2^n-1}) & \dots & f'_{2^n}(\alpha_{2^n-1}) \\ f'_{2^n}(\alpha_{2^n-1}) & f'_3(\alpha_2) + f'_{2^n}(\alpha_{2^n-1}) & \ddots & \vdots \\ \vdots & f'_{2^n}(\alpha_{2^n-1}) & \ddots & f'_{2^n}(\alpha_{2^n-1}) \\ f'_{2^n}(\alpha_{2^n-1}) & \dots & f'_{2^n}(\alpha_{2^n-1}) & f'_{2^n-1}(\alpha_{2^n-2}) + f'_{2^n}(\alpha_{2^n-1}) \end{bmatrix}. \quad (19)$$

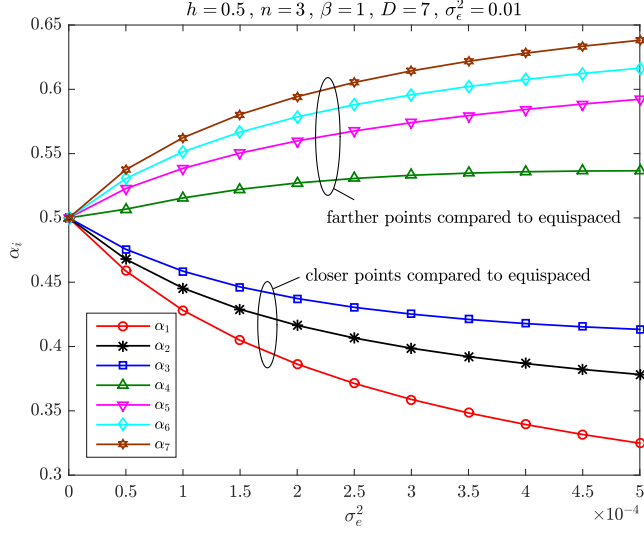


Fig. 2. The values of α_i as a function of σ_e^2 , where $\rho = 2$ and σ_e^2 denotes the variance of the channel estimation error.

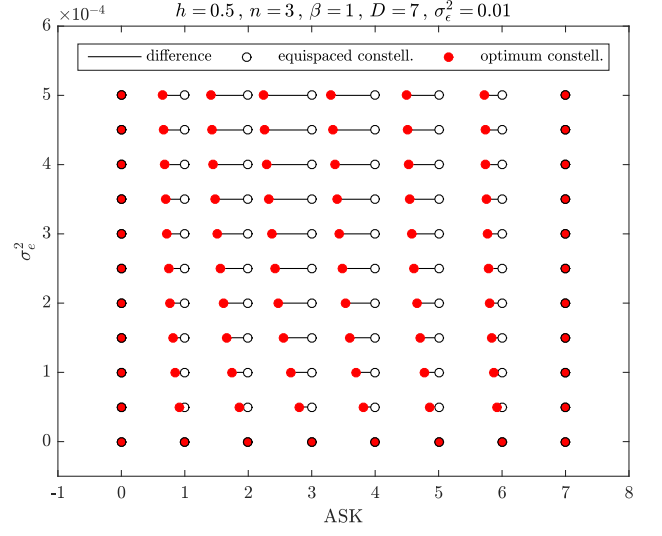


Fig. 3. The 8-ASK constellation for different values of σ_e^2 and $\rho = 2$. For comparison convenience, the equispaced constellation is shown in all steps.

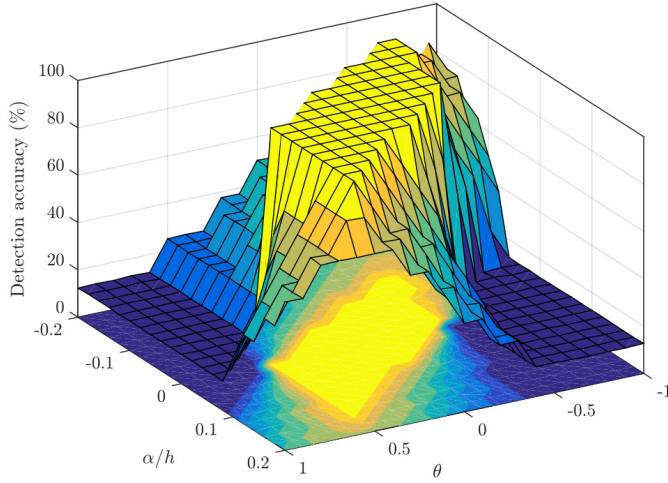


Fig. 4. The detection performance when $NCEE = [\frac{\alpha}{h}, \frac{\alpha}{h}, \dots, \frac{\alpha}{h}]^T$, $\rho = 2$ and $n = 3$. Here, θ is the exact value of the AWGN at the sink.

VI. CONCLUSION

In this work, we proposed a target detection scheme in which independent sensors simultaneously transmitted data on the same frequency band, while detection occurred immediately, within the same time slot. In fact, the introduced approach minimized the detection time to only one time slot which is the fastest possible for any time slot definition. This method has potential applications in specific delay-critical scenarios such as periphery protection and live-time tracking of highly accelerated particles. Among all precodings that allowed for such an instant detection, we studied those which minimized the detection error in the network. The system model described in this work is based on realistic assumptions and is applicable to both wireless and optical systems. However, the hardware technology puts limits on the degree of sensors' synchronization and their dynamic range support for transmission. The system model was investigated in detail,

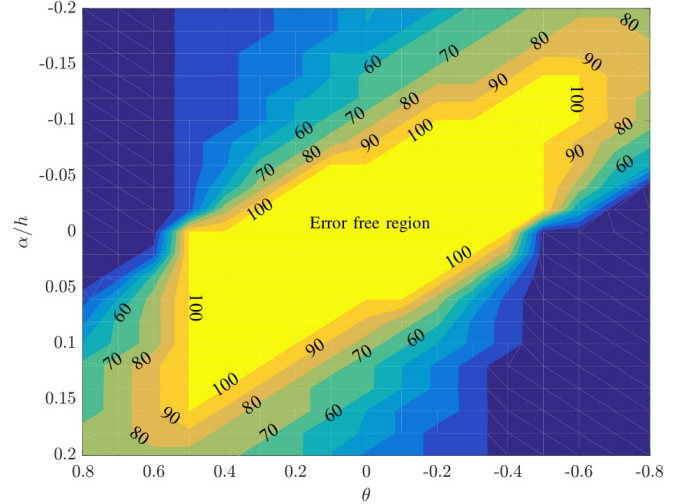


Fig. 5. The contour plot of Figure 4 for $-0.8 \leq \theta \leq 0.8$. The error free region includes all the wide region in the middle.

taking into account the effects of the non-perfect CSI and the AWGN of the receiver. Throughout this work, six major theorems guided the progress of the paper with solid proofs and useful results, and covered a wide range of scenarios for various applications. In order to make the proposed model more understandable, the conceptual connections with other communication techniques were extensively discussed. The importance of the proposed model is highlighted when it is accompanied by conventional frequency division schemes in large scale delay-sensitive networks. The reason is that such networks involve many sensors, and therefore, assigning a unique frequency band to each one is problematic. The combined effects of the channel estimation errors and the noise were both analytically and numerically studied, and a comprehensive evaluation was conducted in the simulations. The results showed that even under the non-perfect CSI and noisy conditions, the system can still be reliable in terms of

high accuracy.

APPENDIX A PROOF OF THEOREM 1

We prove that for any transmitting symbols $\{\lambda_0, \lambda_1\} \subseteq \mathbb{R}$, $\lambda_0 \neq \lambda_1$ which represent the binary decision on a target's presence at each sensor, there is a finite number of $\rho \in \mathbb{R}$ values whose mappings are non-bijective. In other words, we show that if the binary decisions are mapped onto general symbols λ_0 and λ_1 instead of 0 and 1 in \mathbf{b}_t , the number of ρ values for which the signal constellation at the receiver has overlaps is finite. Denoting the new data vector by $\mathbf{c}_t := [c_1^{(t)}, c_2^{(t)}, \dots, c_n^{(t)}]^T$ where $c_i^{(t)} \in \{\lambda_0, \lambda_1\}$ for every $i \in \mathbb{F}_1^n$, the equivalent of Eqs. (3,4) for this case are

$$\beta[1, \dots, 1](\mathbf{v}_t \otimes \mathbf{c}_t) + n_t = \beta \sum_{i=1}^n \rho^{\text{mod}(i-1+t, n)} c_i^{(t)} + n_t. \quad (21)$$

Considering the range of the function $\text{mod}(i-1+t, n)$ and defining the polynomial $f(\rho) := \sum_{i=0}^{n-1} \omega_i \rho^i$ where $\omega_i \in \{0, \lambda_0 - \lambda_1, \lambda_1 - \lambda_0\}$, it can be simply inferred that any overlap of two codewords resulting in a non-bijective mapping can be expressed as a root of function $f(\rho)$. Therefore, the number of non-bijective mappings produced by $\sum_{i=1}^n \rho^{\text{mod}(i-1+t, n)} c_i^{(t)}$ in Eq. (21) will not exceed the number of roots of $f(\rho)$ for all possible definitions of this function. Thus, for every n , there are at most $(n-1)3^n$ values of ρ for which the mapping is not bijective. ■

APPENDIX B PROOF OF THEOREM 2

Assume that for $\rho = \rho_0$, where $\rho_0 \in \mathbb{Q} \setminus \{0, \pm 1\}$, the mapping is not bijective, i.e., either case (I): there exists $\mathbb{U}, \mathbb{V} \subseteq \mathbb{F}_1^n$, such that $\mathbb{U}, \mathbb{V} \neq \emptyset$, $\mathbb{U} \neq \mathbb{V}$, and $\sum_{i \in \mathbb{U}} \rho_0^{i-1} = \sum_{i \in \mathbb{V}} \rho_0^{i-1}$, or case (II): there exists $\mathbb{W} \subseteq \mathbb{F}_1^n$ such that $\mathbb{W} \neq \emptyset$ and $\sum_{i \in \mathbb{W}} \rho_0^{i-1} = 0$. Here in this proof, $a|b$ denotes that integer a divides integer b . First, we consider case (I). Expressing ρ_0 as $\rho_0 = \frac{m}{n}$ where $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, $n \neq 0$ and $\text{gcd}(m, n) = 1$, we have

$$\sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}} m^{i-1}/n^{i-1} = \sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} m^{i-1}/n^{i-1}. \quad (22)$$

Obviously, $\mathbb{I} := (\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}) \cup (\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}) \neq \emptyset$, as otherwise $\mathbb{U} = \mathbb{V}$. So \mathbb{I} has minimum and maximum values, denoted by s and t , respectively. Multiplying Eq. (22) by $m^{1-s}n^{t-1}$ yields

$$\sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}} m^{i-s}n^{t-i} = \sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} m^{i-s}n^{t-i}. \quad (23)$$

Since the exponents in Eq. (23) are all non-negative, and $i = s$ occurs exactly in one side of Eq. (23) as $(\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}) \cap (\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}) = \emptyset$, the relation above implies that $m|n^{t-s}$ and since $\text{gcd}(m, n) = 1$ and $m \in \mathbb{Z}_{\geq 0}$, we have $m = 1$. Thus,

$$\sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}} n^{t-i} = \sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} n^{t-i}. \quad (24)$$

For simplicity and without loss of generality, assume that $t \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}$. Thus, Eq. (24) yields

$$\begin{aligned} 1 + \sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}, i \neq t} n^{t-i} &= n \sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} n^{t-i-1} \\ \Rightarrow n \left(\underbrace{\sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} n^{t-i-1} - \sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}, i \neq t} n^{t-i-1}}_{\in \mathbb{Z}} \right) &= 1. \end{aligned} \quad (25)$$

As a result, $n|1$ which implies that $\rho_0 = \pm 1$. This contradicts the primary assumption $\rho_0 \in \mathbb{Q} \setminus \{0, \pm 1\}$. Now, let's assume the second case occurs. Expressing ρ_0 in terms of m and n as in case (I), $\sum_{i \in \mathbb{W}} \rho_0^{i-1} = 0$ yields $\sum_{i \in \mathbb{W}} m^{i-1}n^{W-i} = 0$, where $W := \max\{\mathbb{W}\}$. This means that $m^{W-1} = -\sum_{i \in \mathbb{W} \setminus W} m^{i-1}n^{W-i}$ or $m^{W-w} = -\sum_{i \in \mathbb{W} \setminus W} m^{i-w}n^{W-i}$ where $w := \min\{\mathbb{W}\}$. If $|\mathbb{W}| = 1$, we obtain $\rho_0 = 0$ which contradicts $\rho_0 \in \mathbb{Q} \setminus \{0, \pm 1\}$. Thus, assuming $|\mathbb{W}| > 1$, the obtained relation implies that $m|m^{W-w} = -\sum_{i \in \mathbb{W} \setminus W} m^{i-w}n^{W-i} = -n^{W-w} - m \sum_{i \in \mathbb{W} \setminus W, w} m^{i-w-1}n^{W-i}$. Thus, $m|n^{W-w}$ and since $\text{gcd}(m, n) = 1$ and $m \in \mathbb{Z}_{\geq 0}$, we obtain $m = 1$. Substituting this value into the recent equation yields $1 = -n^{W-w} - \sum_{i \in \mathbb{W} \setminus W, w} n^{W-i}$. This implies that $n|1$. Therefore, $\rho_0 = \pm 1$, and the assertion follows by contradiction. ■

APPENDIX C PROOF OF THEOREM 3

Defining the probability density function (PDF) of the noise at the sink node as $f(\lambda) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\lambda^2}{2\sigma^2}}$, and denoting the constellation point by m_i , the probability (Pr) of error in detection, denoted by Pr_E , can be expressed as follows

$$Pr_E = 1 - \sum_{i=1}^{2^n} Pr(m_i + \lambda \in \mathbb{M}_i | m_i) Pr(m_i) \quad (26)$$

$$\begin{aligned} &= 1 - \sum_{i=1}^{2^n} Pr(\lambda \in \mathbb{M}_i - m_i) \prod_{j=1}^n \frac{1}{2} \\ &= 1 - \frac{1}{2^n} \left(\sum_{i=1}^{2^n-2} \int_{-\alpha_i}^{\alpha_{i+1}} f(\lambda) d\lambda + \int_{-\infty}^{\alpha_1} f(\lambda) d\lambda + \int_{-\alpha_{2^n-1}}^{\infty} f(\lambda) d\lambda \right), \end{aligned} \quad (27)$$

where $\mathbb{M}_1 := (-\infty, m_1 + \alpha_1]$, $\mathbb{M}_i := (m_i - \alpha_{i-1}, m_i + \alpha_i] \forall i \in \mathbb{F}_2^{2^n-1}$, $\mathbb{M}_{2^n} := (m_{2^n} - \alpha_{2^n-1}, +\infty)$, and α_i s are positive variables as shown in Figure 6. Thus, the problem is

$$\text{maximize: } \sum_{\alpha_1, \dots, \alpha_{2^n-1}} \sum_{i=1}^{2^n-2} \int_{-\alpha_i}^{\alpha_{i+1}} f(\lambda) d\lambda + \int_{-\infty}^{\alpha_1} f(\lambda) d\lambda + \int_{-\alpha_{2^n-1}}^{\infty} f(\lambda) d\lambda \quad (28)$$

$$\text{subject to: } 2 \sum_{i=1}^{2^n-1} \alpha_i = D \text{ (note that } D := m_{2^n} - m_1) \quad (29)$$

or equivalently

$$\text{maximize: } \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2^n-2}} \sum_{i=1}^{2^n-2} \int_0^{\alpha_i} f(\lambda) d\lambda + \int_0^{\frac{D}{2} - \sum_{j=1}^{2^n-2} \alpha_j} f(\lambda) d\lambda, \quad (30)$$

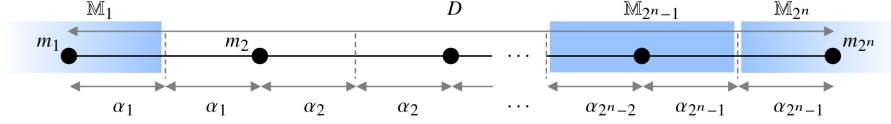


Fig. 6. 2^n -ASK in a general form where the constellation points are spaced by α_i values. Here, the fixed parameter D is the maximum Euclidean distance.

$$\text{subject to: } \sum_{i=1}^{2^n-2} \alpha_i \leq \frac{D}{2}, \alpha_i \geq 0 \forall i \in \mathbb{F}_1^{2^n-2} \quad (31)$$

where Eq. (30) was derived from Eqs. (28,29) by eliminating the dependent α_{2^n-1} , excluding the constant terms and using

$$\int_{-\alpha_i}^{\alpha_{i+1}} f(\lambda) d\lambda = \int_0^{\alpha_i} f(\lambda) d\lambda + \int_0^{\alpha_{i+1}} f(\lambda) d\lambda \quad (32)$$

which holds for even functions. Denoting the objective function in Eq. (30) by function g and using α_{2^n-1} instead of $\frac{D}{2} - \sum_{j=1}^{2^n-2} \alpha_j$ for compactness, the Hessian is expressed in Eq. (33). Thus, defining $\mathbf{z} := [z_1, z_2, \dots, z_{2^n-2}]^T \in \{\mathbb{R}^{2^n-2} \setminus \{0\}\}$, and considering that $f'(\lambda) = \frac{-\lambda}{\sigma^2 \sqrt{2\pi\sigma^2}} e^{\frac{-\lambda^2}{2\sigma^2}} < 0$ for $\lambda > 0$, one can simply prove that for every $\mathbf{z} \in \{\mathbb{R}^{2^n-2} \setminus \{0\}\}$,

$$\mathbf{z}^T \nabla^2 g \mathbf{z} = \sum_{i=1}^{2^n-2} f'(\alpha_i) z_i^2 + f'(\alpha_{2^n-1}) \left(\sum_{i=1}^{2^n-2} z_i \right)^2 < 0. \quad (34)$$

Therefore, $\nabla^2 g$ is negative definite ($\nabla^2 g \prec 0$), i.e., g is maximized when $\nabla g = \mathbf{0}$. $\forall j \in \mathbb{F}_1^{2^n-2} : \frac{\partial g}{\partial \alpha_j} = f(\alpha_j) - f(\alpha_{2^n-1}) = 0 \iff \alpha_1 = \alpha_2 = \dots = \alpha_{2^n-1} = \frac{D}{2^{n+1}-2}$. ■

APPENDIX D PROOF OF THEOREM 4

Remark 1. The procedure in this proof will be used to prove Theorem 6 in Appendix F.

Definition 1. For every nonempty set $\mathbb{X} \subseteq \mathbb{R}$ and every $y, z \in \mathbb{R}$, $y + z\mathbb{X} := \{y + zx | x \in \mathbb{X}\}$.

For our problem, an EBM constellation can be formulated as $\delta + \eta \mathbb{F}_0^{2^n-1}$, where $\delta \in \mathbb{R}$ and $\eta \in \mathbb{R}_{>0}$. Since OOK is used at the transmitter, $0 \in \delta + \eta \mathbb{F}_0^{2^n-1}$. Therefore, $\exists P \in \mathbb{F}_0^{2^n-1} \ni \delta = -P\eta$, which means $\delta + \eta \mathbb{F}_0^{2^n-1} = \eta \mathbb{F}_{-P}^{2^n-1-P}$. Thus, in order to prove the necessary condition, it is sufficient to show that if $\eta \mathbb{F}_M^{2^n+M-1} = \{0\} \cup \mathbb{S}(\mathbb{F}_1^n)$ where $M \in \mathbb{F}_{1-2^n}^n$, $\eta \in \mathbb{R}_{>0}$ and for every nonempty set $\mathbb{X} \subseteq \mathbb{F}_1^n$, $\mathbb{S}(\mathbb{X})$ is defined on the amplitude set $\{a_1, a_2, \dots, a_n\} \subset \mathbb{R}$ as

$$\mathbb{S}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} a_i \mid \mathbb{Y} \subseteq \mathbb{X}, \mathbb{Y} \neq \emptyset \right\}, \quad (35)$$

then $\{a_i | i \in \mathbb{F}_1^n\} = \{\eta(-1)^{m_i} 2^{i-1} | i \in \mathbb{F}_1^n\}$, where $m_1, m_2, \dots, m_n \in \{0, 1\}$ are such that the binary representation of $-M$ is $-M = (m_n m_{n-1} \dots m_1)_2$. The proof is as follows. Without loss of generality, let's assume that $a_1 < a_2 < \dots < a_n$. Thus, $\eta M = \min\{\eta \mathbb{F}_M^{2^n+M-1}\} =$

$\min\{\{0\} \cup \mathbb{S}(\mathbb{F}_1^n)\}$ and therefore, it holds that

$$\text{if } M \neq 0 \text{ then } \exists L \in \mathbb{F}_1^n \text{ such that } M = \frac{1}{\eta} \sum_{i=1}^L a_i, \quad (36)$$

$$a_i < 0 \text{ for all } i \in \mathbb{F}_1^L \text{ and } a_i > 0 \text{ for all } i \in \mathbb{F}_{L+1}^n.$$

Definition 2. For any set \mathbb{X} , $|\mathbb{X}|$ denotes the numbers of elements in \mathbb{X} with *distinct numerical values*, when each member of the set takes its corresponding numerical value, while $|\mathbb{X}|_S$ is the number of *distinct symbolic elements* in \mathbb{X} , regardless of their corresponding numerical values, where symbolic elements $x_1, x_2 \in \mathbb{X}$, $x_1 := \sum_{i \in \mathbb{W}_1} b_i$, $x_2 := \sum_{i \in \mathbb{W}_2} b_i$ are called distinct, if $\mathbb{W}_1 \neq \mathbb{W}_2$. For example, if $b_1 = b_2 = 1$, $b_3 = 2$ and $\mathbb{X} = \{b_1, b_2, b_1 + b_2, b_2 + b_1, b_3, b_3\}$, we have $|\mathbb{X}| = 2$ as values of \mathbb{X} are $\{1, 2\}$, and $|\mathbb{X}|_S = 4$, as symbolic elements of \mathbb{X} are $\{b_1, b_2, b_1 + b_2, b_3\}$.

Corollary D.1. $\mathbb{T}(\mathbb{F}_1^n) = \mathbb{F}_1^{2^n-1}$, where, for every nonempty subset \mathbb{X} of \mathbb{F}_1^n , $\mathbb{T}(\mathbb{X})$ is defined as

$$\mathbb{T}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} \frac{|a_i|}{\eta} \mid \mathbb{Y} \subseteq \mathbb{X}, \mathbb{Y} \neq \emptyset \right\}. \quad (37)$$

Proof D.1: It is clear that $a_i \neq 0 \forall i \in \mathbb{F}_1^n$, as otherwise $|\eta \mathbb{F}_M^{2^n+M-1}| = 2^n \neq |\{0\} \cup \mathbb{S}(\mathbb{F}_1^n)|$. On the other hand, since for every $i \in \mathbb{F}_1^n$, $a_i \in \{0\} \cup \mathbb{S}(\mathbb{F}_1^n) = \eta \mathbb{F}_M^{2^n+M-1}$, we conclude that $\mathbb{T}(\mathbb{F}_1^n) \subseteq \mathbb{Z}$, and since $\mathbb{T}(\mathbb{F}_1^n) \subseteq \mathbb{R}_{\geq 0}$, we have $\min\{\mathbb{T}(\mathbb{F}_1^n)\} \geq 1$. Also $\max\{\mathbb{T}(\mathbb{F}_1^n)\} = \frac{1}{\eta} \sum_{i=1}^n |a_i|$,

$$\frac{1}{\eta} \sum_{i=1}^n |a_i| = -\frac{1}{\eta} \sum_{i=1}^L a_i + \frac{1}{\eta} \sum_{i=L+1}^n a_i \quad (38)$$

$$= -M + \frac{1}{\eta} \max\{\{0\} \cup \mathbb{S}(\mathbb{F}_1^n)\} \quad (39)$$

$$= -M + \frac{1}{\eta} \max\{\eta \mathbb{F}_M^{2^n+M-1}\} = 2^n - 1, \quad (40)$$

$$\Rightarrow \mathbb{T}(\mathbb{F}_1^n) \subseteq \mathbb{F}_1^{2^n-1}. \quad (41)$$

Corollary D.1.1. $|\mathbb{T}(\mathbb{F}_1^n)|_S = |\mathbb{T}(\mathbb{F}_1^n)|$. In other words, if $u := \frac{1}{\eta} \sum_{i \in \mathbb{U}} |a_i|$ and $v := \frac{1}{\eta} \sum_{i \in \mathbb{V}} |a_i|$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{F}_1^n$, $\mathbb{U}, \mathbb{V} \neq \emptyset$, and $\mathbb{U} \neq \mathbb{V}$, then $u \neq v$.

Proof D.1.1: If $u = v$, then $\sum_{i \in \mathbb{U}} |a_i| = \sum_{i \in \mathbb{V}} |a_i|$, so

$$\sum_{i \in \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}} |a_i| = \sum_{i \in \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}} |a_i| \quad (42)$$

where, due to $(\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\}) \cap (\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\}) = \emptyset$, the equal terms in both sides have been removed. Equation (42) can be written as $\sum_{i \in \mathbb{A}} a_i = \sum_{i \in \mathbb{B}} a_i$, where

$$\mathbb{A} := (\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} \cap \mathbb{F}_{L+1}^n) \cup (\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} \cap \mathbb{F}_1^L) \quad (43)$$

$$\mathbb{B} := (\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} \cap \mathbb{F}_{L+1}^n) \cup (\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} \cap \mathbb{F}_1^L). \quad (44)$$

From the definitions above, one can simply prove that $\mathbb{A} \cap \mathbb{B} = \emptyset$. Further, we note that $\mathbb{A} \neq \mathbb{B}$. To prove this, let's assume $\mathbb{A} = \mathbb{B}$. From $\mathbb{A} \cap \mathbb{B} = \emptyset$, we obtain $\mathbb{A} = \mathbb{B} = \emptyset$, which yields

$$\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} \cap \mathbb{F}_{L+1}^n = \emptyset, \quad (45)$$

$$\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} \cap \mathbb{F}_1^L = \emptyset, \quad (46)$$

$$\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} \cap \mathbb{F}_{L+1}^n = \emptyset, \quad (47)$$

$$\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} \cap \mathbb{F}_1^L = \emptyset. \quad (48)$$

From Eqs. (45,48), we have $\mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} \cap \mathbb{F}_1^n = \mathbb{U} \cap \{\mathbb{F}_1^n \setminus \mathbb{V}\} = \emptyset$ which means $\mathbb{U} \subseteq \mathbb{V}$. Similarly, from Eqs. (46,47), we have $\mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} \cap \mathbb{F}_1^n = \mathbb{V} \cap \{\mathbb{F}_1^n \setminus \mathbb{U}\} = \emptyset$ which means $\mathbb{V} \subseteq \mathbb{U}$. Therefore, we infer $\mathbb{U} = \mathbb{V}$ which contradicts the assumption of Corollary D.1.1. As a result of this part, we have $\mathbb{A} \neq \mathbb{B}$. On the other hand, since $\mathbb{A}, \mathbb{B} \subseteq \mathbb{F}_1^n$, the result $\mathbb{A} \neq \mathbb{B}$ yields $\sum_{i \in \mathbb{A}} a_i, \sum_{i \in \mathbb{B}} a_i \in \{0\} \cup \mathbb{S}(\mathbb{F}_1^n)$ and $\sum_{i \in \mathbb{A}} a_i \neq \sum_{i \in \mathbb{B}} a_i$, as otherwise $|\eta \mathbb{F}_M^{2^n+M-1}| \neq |\{0\} \cup \mathbb{S}(\mathbb{F}_1^n)|$. This result contradicts $\sum_{i \in \mathbb{A}} a_i = \sum_{i \in \mathbb{B}} a_i$ and the assertion follows. \square

From Corollary D.1.1, Eq. (41), and the fact that in terms of symbolic elements generated by set function $\mathbb{T}(\mathbb{X})$, we have $|\mathbb{T}(\mathbb{F}_1^n)|_S = 2^n - 1$, it is concluded that $\mathbb{T}(\mathbb{F}_1^n) = \mathbb{F}_1^{2^n-1}$ and Corollary D.1 follows. \blacksquare

Defining

$$\{c_i | i \in \mathbb{F}_1^n\} := \{|a_i|/\eta | i \in \mathbb{F}_1^n\}, \quad (49)$$

where $0 < c_1 < c_2 < \dots < c_n$

and a summation on this set for every $\mathbb{X} \subseteq \mathbb{F}_1^n$ as

$$\mathbb{D}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} c_i \mid \mathbb{Y} \subseteq \mathbb{X}, \mathbb{Y} \neq \emptyset \right\}, \quad (50)$$

it is clear that $\mathbb{T}(\mathbb{F}_1^n) = \mathbb{D}(\mathbb{F}_1^n)$. Thus, from Corollary D.1, we have $\mathbb{D}(\mathbb{F}_1^n) = \mathbb{F}_1^{2^n-1}$.

Corollary D.2. $\forall i \in \mathbb{F}_1^n : c_i = 2^{i-1}$.

Proof D.2: For $i = 1$, we have $c_1 = \min\{\mathbb{D}(\mathbb{F}_1^n)\} = \min\{\mathbb{F}_1^{2^n-1}\} = 2^{1-1}$ and for $i = 2$, we have $c_2 = \min\{\mathbb{D}(\mathbb{F}_1^n) \setminus c_1\} = \min\{\mathbb{F}_1^{2^n-1} \setminus 1\} = 2^{2-1}$. Now let's assume that for every $m \in \mathbb{F}_1^k$ where $k \leq n-1$, the relation $c_m = 2^{m-1}$ holds. It suffices to prove that $c_{k+1} = 2^k$.

Corollary D.2.1. $\forall x \in \{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\} : x \geq c_{k+1}$.

Proof D.2.1: Assume that $\exists x \in \{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\}$ such that $x < c_{k+1}$. Thus, $x \in \mathbb{D}(\mathbb{F}_1^n)$ and $x \notin \mathbb{D}(\mathbb{F}_1^k)$. From $x \in \mathbb{D}(\mathbb{F}_1^n)$, it yields that $\exists \alpha_1, \alpha_2, \dots, \alpha_n \in \{0, 1\}$ such that $x = \sum_{i=1}^n \alpha_i c_i$, and from $x \notin \mathbb{D}(\mathbb{F}_1^k)$, it is concluded that $\exists j > k \ni \alpha_j \neq 0$. Hence, x can be expressed as $x = c_j + \Delta$, where $\Delta \geq 0$. Therefore, since sequence $(c_i)_{i=1}^n$ is increasing, we obtain the relation $c_{k+1} \leq c_j \leq x < c_{k+1}$ which contradicts our assumption. \square

Corollary D.2.2. $c_{k+1} \in \{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\}$.

Proof D.2.2: It is clear that $c_{k+1} \in \mathbb{D}(\mathbb{F}_1^n)$, so we prove that

$c_{k+1} \notin \mathbb{D}(\mathbb{F}_1^k)$. If $c_{k+1} \in \mathbb{D}(\mathbb{F}_1^k)$, then it can be written as a sum of several c_i s, where $i \in \mathbb{F}_1^k$. This implies that $|\mathbb{D}(\mathbb{F}_1^k)| < 2^n - 1$, as one member is repetitive. On the other hand $|\mathbb{D}(\mathbb{F}_1^k)| = |\mathbb{F}_1^{2^k-1}| = 2^k - 1$, which contradicts the assumption we made. \square

From the assertions above, it is deduced that

$$c_{k+1} = \min\{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\} = \min\{\mathbb{F}_1^{2^n-1} \setminus \mathbb{D}(\mathbb{F}_1^k)\}. \quad (51)$$

On the other hand, according to the assumption $c_m = 2^{m-1} \forall m \in \mathbb{F}_1^k$, we have $\min\{\mathbb{D}(\mathbb{F}_1^k)\} = 1$ and $\max\{\mathbb{D}(\mathbb{F}_1^k)\} = \sum_{i=1}^k 2^{i-1} = 2^k - 1$. Also similar to Corollary D.1.1, since the symbolic function $\sum_{i \in \mathbb{Y}} c_i$ in the definition of $\mathbb{D}(\mathbb{X})$ returns distinct numerical values when $\mathbb{X} = \mathbb{F}_1^n$, as otherwise $\mathbb{D}(\mathbb{F}_1^n) \neq \mathbb{F}_1^{2^n-1}$, the symbolic set $\mathbb{D}(\mathbb{F}_1^k)$ should also return distinct numerical values, as $\mathbb{F}_1^k \subseteq \mathbb{F}_1^n$. This means that $|\mathbb{D}(\mathbb{F}_1^k)| = |\mathbb{D}(\mathbb{F}_1^k)|_S = 2^k - 1$. Thus, from $\min\{\mathbb{D}(\mathbb{F}_1^k)\} = 1$, $\max\{\mathbb{D}(\mathbb{F}_1^k)\} = 2^k - 1$, and $|\mathbb{D}(\mathbb{F}_1^k)| = 2^k - 1$, we infer $\mathbb{D}(\mathbb{F}_1^k) = \mathbb{F}_1^{2^k-1}$. Hence, $c_{k+1} = \min\{\mathbb{F}_1^{2^n-1} \setminus \mathbb{D}(\mathbb{F}_1^k)\} = \min\{\mathbb{F}_1^{2^n-1} \setminus \mathbb{F}_1^{2^k-1}\} = 2^k$ and therefore, Corollary D.2 follows by *strong* induction. \blacksquare

As a result of Corollary D.2 and Definition (49), we have

$$\bigcup_{i=1}^n \{\eta 2^{i-1}\} = \bigcup_{i=1}^n \{|a_i|\} = \left(\bigcup_{i=1}^L \{-a_i\} \right) \cup \left(\bigcup_{i=L+1}^n \{a_i\} \right). \quad (52)$$

Considering the fact that there is only one binary representation for any non-negative integer $-M$, Eq. (52) implies that $\{a_i | i \in \mathbb{F}_1^n\} = \{\eta(-1)^{m_i} 2^{i-1} | i \in \mathbb{F}_1^n\}$, where $-M = (m_n m_{n-1} \dots m_1)_2$. So far, Theorem 4 has been proved for $M \in \mathbb{F}_{1-2^n}^{-1}$. For the case of $M = 0$, it is obvious that all a_i values must be positive and there is no combination of a_i s which equals M . Thus, by eliminating the zero member from both sides of $\eta \mathbb{F}_M^{2^n+M-1} = \{0\} \cup \mathbb{S}(\mathbb{F}_1^n)$, the problem is converted to the previous case (starting from Corollary D.1) and furthermore, we have $\mathbb{S}(\mathbb{X}) = \eta \mathbb{T}(\mathbb{X}) = \eta \mathbb{D}(\mathbb{X})$. This completes the proof of the necessary condition. \blacksquare

For the case of sufficient condition, we should prove that if $\{a_i | i \in \mathbb{F}_1^n\} = \{\eta(-1)^{m_i} 2^{i-1} | i \in \mathbb{F}_1^n\}$, where $-M = (m_n m_{n-1} \dots m_1)_2$, then $\{0\} \cup \mathbb{S}(\mathbb{F}_1^n) = \eta \mathbb{F}_M^{2^n+M-1}$. For $n = 1$, we have $M = 0$ or $M = -1$. If $M = 0$, $\{0\} \cup \mathbb{S}(\mathbb{F}_1^n) = \{0, \eta\} = \eta \mathbb{F}_M^{2^n+M-1}$, and if $M = -1$, $\{0\} \cup \mathbb{S}(\mathbb{F}_1^n) = \{0, -\eta\} = \eta \mathbb{F}_M^{2^n+M-1}$. Thus, the assertion holds for $n = 1$. Similarly, for $n = 2$, one can simply show that $\{0\} \cup \mathbb{S}(\mathbb{F}_1^n) = \eta \mathbb{F}_M^{2^n+M-1}$ holds for all $M \in \mathbb{F}_{1-2^k}^0$. Now let's assume the assertion holds for $n = k$ and all $M \in \mathbb{F}_{1-2^k}^0$ where $-M = (m_k m_{k-1} \dots m_1)_2$. It suffices to prove that it also holds for $n = k+1$ and every $M' \in \mathbb{F}_{1-2^{k+1}}^0$, where $-M' = (m'_{k+1} m'_k \dots m'_1)_2$. The proof is as follows. Since

$$\mathbb{E} := \{a_i | i \in \mathbb{F}_1^{k+1}\} = \{\eta(-1)^{m'_i} 2^{i-1} | i \in \mathbb{F}_1^{k+1}\}, \quad (53)$$

$$\nabla^2 g = \begin{bmatrix} f'(\alpha_1) + f'(\alpha_{2^n-1}) & f'(\alpha_{2^n-1}) & \dots & f'(\alpha_{2^n-1}) \\ f'(\alpha_{2^n-1}) & f'(\alpha_2) + f'(\alpha_{2^n-1}) & \ddots & \vdots \\ \vdots & f'(\alpha_{2^n-1}) & \ddots & f'(\alpha_{2^n-1}) \\ f'(\alpha_{2^n-1}) & \dots & f'(\alpha_{2^n-1}) & f'(\alpha_{2^n-2}) + f'(\alpha_{2^n-1}) \end{bmatrix}. \quad (33)$$

we have $\{\mathbb{E} \setminus \eta(-1)^{m'_{k+1}} 2^k\} = \{\eta(-1)^{m'_i} 2^{i-1} | i \in \mathbb{F}_1^k\}$. Thus, the assumption we made for $n = k$ yields $\{0\} \cup \mathbb{G} = \eta \mathbb{F}_{M''}^{2^k + M'' - 1}$, where $\mathbb{G} := \{\sum_{i \in \mathbb{Y}} g_i | \mathbb{Y} \subseteq \mathbb{F}_1^k, \mathbb{Y} \neq \emptyset\}$,

$$\bigcup_{i=1}^k \{g_i\} := \{\mathbb{E} \setminus a_{ex} | a_{ex} := \eta(-1)^{m'_{k+1}} 2^k, ex \in \mathbb{F}_1^{k+1}\}, \quad (54)$$

and $-M'' := \overline{(m'_k m'_{k-1} \dots m'_1)_2}$. From $M' \in \mathbb{F}_{1-2^{k+1}}^0$, $M'' \in \mathbb{F}_{1-2^k}^0$, the relation $M'' = M' + m'_{k+1} 2^k$, and the fact that $m'_{k+1} \in \{0, 1\}$, the relations in Eqs. (55-59) hold and the assertion follows by induction. ■

APPENDIX E PROOF OF THEOREM 5

We prove that for some pairs of (n_T, n) , $n_T < n$, there exists an EBM, whereas for some other pairs, such a mapping does not exist. Corollaries E.1 and E.2 provide a wide range of solutions for each case.

Corollary E.1. *There exists no EBM for all pairs of the form*

$$(n_T, n) = \left(\overline{(t s t s_{-1} \dots t_1 0)_2}, \overline{(v s v s_{-1} \dots v_1 1)_2} \right), \quad (60)$$

where $v_i, t_i \in \{0, 1\}$ and $S \in \mathbb{N}$ are such that $n_T < n$ and

$$\exists m \in \mathbb{F}_1^S \ni t_m = 1 \text{ and } v_m = 0. \quad (61)$$

Proof E.1: Let's assume that such a mapping exists. Defining

$$\mathbb{S}_{n_T}(\mathbb{F}_1^n) := \left\{ \sum_{i \in \mathbb{Y}} a_i \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\} \quad (62)$$

analogous to Appendix D, there exists $\delta \in \mathbb{R}$ such that $\mathbb{S}_{n_T}(\mathbb{F}_1^n) = \delta + \eta \mathbb{F}_0^{n_c - 1}$, where $n_c := \binom{n}{n_T}$ and $\eta \in \mathbb{R}_{>0}$. Expressing $\frac{1}{\eta}(a_i - \frac{\delta}{n_T})$ as $\frac{1}{\eta}(a_i - \frac{\delta}{n_T}) = \lfloor \frac{1}{\eta}(a_i - \frac{\delta}{n_T}) \rfloor + p_i$ for every $i \in \mathbb{F}_1^n$, where $\lfloor x \rfloor$ denotes the floor function and $0 \leq p_i < 1 \forall i \in \mathbb{F}_1^n$, and assuming that $j, k \in \mathbb{F}_1^n$ are two arbitrarily chosen indices, we first prove that $p_j = p_k$. The proof is as follows. Since $n_T < n$, there exists a set $\mathbb{W} \subseteq \mathbb{F}_1^n$, such that $|\mathbb{W}| = n_T - 1$, and $j, k \notin \mathbb{W}$. Defining $\mathbb{J} := \mathbb{W} \cup \{j\}$ and $\mathbb{K} := \mathbb{W} \cup \{k\}$, it is clear that $\sum_{i \in \mathbb{J}} a_i, \sum_{i \in \mathbb{K}} a_i \in \mathbb{S}_{n_T}(\mathbb{F}_1^n) = \delta + \eta \mathbb{F}_0^{n_c - 1}$. As a result, we have $\sum_{i \in \mathbb{J}} \frac{1}{\eta}(a_i - \frac{\delta}{n_T}), \sum_{i \in \mathbb{K}} \frac{1}{\eta}(a_i - \frac{\delta}{n_T}) \in \mathbb{Z}$. From $\sum_{i \in \mathbb{J}} \frac{1}{\eta}(a_i - \frac{\delta}{n_T}) \in \mathbb{Z}$, the relations below follow

$$\sum_{i \in \mathbb{J}} \frac{a_i n_T - \delta}{\eta n_T} = \left(\sum_{i \in \mathbb{J}} \lfloor \frac{a_i n_T - \delta}{\eta n_T} \rfloor + \sum_{i \in \mathbb{J}} p_i \right) \in \mathbb{Z} \quad (63)$$

$$\Rightarrow \sum_{i \in \mathbb{J}} p_i = \left(\sum_{i \in \mathbb{J}} \lfloor \frac{a_i n_T - \delta}{\eta n_T} \rfloor + p_j \right) \in \mathbb{Z} \quad (64)$$

$$\Rightarrow \left(\sum_{i \in \mathbb{J}} \lfloor \frac{a_i n_T - \delta}{\eta n_T} \rfloor - \lfloor \sum_{i \in \mathbb{J}} \frac{a_i n_T - \delta}{\eta n_T} \rfloor + p_j \right) \in \mathbb{Z} \quad (65)$$

$$\Rightarrow \left(\sum_{i \in \mathbb{W}} p_i - \lfloor \sum_{i \in \mathbb{W}} p_i \rfloor + p_j \right) \in \mathbb{Z} \quad (66)$$

Since $\sum_{i \in \mathbb{W}} p_i - \lfloor \sum_{i \in \mathbb{W}} p_i \rfloor \in [0, 1)$ and $0 \leq p_j < 1$, we obtain

$$0 \leq \sum_{i \in \mathbb{W}} p_i - \lfloor \sum_{i \in \mathbb{W}} p_i \rfloor + p_j < 2. \quad (67)$$

Relations (66) and (67) imply that

$$p_j = \begin{cases} 0 & \text{if } \sum_{i \in \mathbb{W}} p_i \in \mathbb{Z}; \\ 1 - \sum_{i \in \mathbb{W}} p_i + \lfloor \sum_{i \in \mathbb{W}} p_i \rfloor & \text{else.} \end{cases} \quad (68)$$

Similarly, it can be proved that

$$p_k = \begin{cases} 0 & \text{if } \sum_{i \in \mathbb{W}} p_i \in \mathbb{Z}; \\ 1 - \sum_{i \in \mathbb{W}} p_i + \lfloor \sum_{i \in \mathbb{W}} p_i \rfloor & \text{else.} \end{cases} \quad (69)$$

Equations (68,69) imply that $p_j = p_k$. As a result, $p := p_1 = p_2 = \dots = p_n$. Defining $K := \sum_{i \in \mathbb{J}} p_i = |\mathbb{J}|p = n_T p$, we obtain $p = \frac{K}{n_T}$. Note that according to Eq. (64), $K \in \mathbb{Z}$. Thus, Eq. (62) and the assumption $\mathbb{S}_{n_T}(\mathbb{F}_1^n) = \delta + \eta \mathbb{F}_0^{n_c - 1}$ yield

$$\mathbb{F}_0^{n_c - 1} = \frac{1}{\eta} (\mathbb{S}_{n_T}(\mathbb{F}_1^n) - \delta) \quad (70)$$

$$= \left\{ \sum_{i \in \mathbb{Y}} \frac{a_i n_T - \delta}{\eta n_T} \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\} \quad (71)$$

$$= \left\{ \sum_{i \in \mathbb{Y}} \left(\lfloor \frac{a_i n_T - \delta}{\eta n_T} \rfloor + \frac{K}{n_T} \right) \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\} \quad (72)$$

$$= \left\{ K + \sum_{i \in \mathbb{Y}} \left\lfloor \frac{a_i n_T - \delta}{\eta n_T} \right\rfloor \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\} \quad (73)$$

$$= K + \left\{ \sum_{i \in \mathbb{Y}} \left\lfloor \frac{a_i n_T - \delta}{\eta n_T} \right\rfloor \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\}. \quad (74)$$

From $\mathbb{F}_0^{n_c - 1} - K = \mathbb{F}_{-K}^{-K + n_c - 1}$ and Eqs. (71,74), we have

$$\left\{ \sum_{i \in \mathbb{Y}} \left\lfloor \frac{a_i n_T - \delta}{\eta n_T} \right\rfloor \mid \mathbb{Y} \subseteq \mathbb{F}_1^n, |\mathbb{Y}| = n_T \right\} = \mathbb{F}_{-K}^{-K + n_c - 1}. \quad (75)$$

Thus, by adding up the members of the sets in both sides of Eq. (75) and considering the fact that the number of repetitions for each particular $\lfloor \frac{1}{\eta}(a_i - \frac{\delta}{n_T}) \rfloor$ is $\binom{n-1}{n_T-1}$, the relations in Eqs. (76-83) are concluded ($a|b$ means a divides b). On the other hand, from the Lucas' theorem [23] for prime modulus two, we have

$$\binom{n-1}{n_T} \stackrel{2}{=} \binom{0}{0} \prod_{i=1}^S \binom{v_i}{t_i} \quad (84)$$

where t_i s and v_i s were defined in Eqs. (60,61). Now, if we apply the condition in Eq. (61) on Eq. (84), we get

$$\binom{n-1}{n_T} \stackrel{2}{=} \prod_{i=1}^S \binom{v_i}{t_i} \stackrel{2}{=} \underbrace{\binom{v_m}{t_m}}_{=0} \prod_{\substack{i=1 \\ i \neq m}}^S \binom{v_i}{t_i} \stackrel{2}{=} 0 \quad (85)$$

which contradicts Eq. (83) and the assertion follows. ■

Corollary E.2. *For all pairs of the form $(n_T, n) = (1, n)$ or $(n_T, n) = (n-1, n)$, an EBM always exists, regardless of the value of n .*

Proof E.2: Considering the definitions in Corollary E.1, for $n_T = 1$, the existence of an EBM is obvious, as due to $\mathbb{S}_1(\mathbb{F}_1^n) = \{a_1, a_2, \dots, a_n\}$, the values of a_i s which fulfill the EBM condition $\mathbb{S}_1(\mathbb{F}_1^n) = \delta + \eta \mathbb{F}_0^{n_c - 1}$ are already given by this formula. Now, we consider the case $n_T = n-1$. Assuming $\{a_1, a_2, \dots, a_n\} = \frac{\delta_0}{n_T} + \eta_0 \mathbb{F}_0^{n-1}$ where $\eta_0 \in \mathbb{R}_{>0}$ and $\delta_0 \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{S}_{n-1}(\mathbb{F}_1^n) &= (\sum_{i=1}^n a_i) - \mathbb{S}_1(\mathbb{F}_1^n) \\ &= (n_T + 1) \frac{\delta_0}{n_T} + \left(\eta_0 \sum_{i=0}^{n-1} i \right) - \left(\frac{\delta_0}{n_T} + \eta_0 \mathbb{F}_0^{n-1} \right) \end{aligned} \quad (86)$$

$$= \delta_0 + \eta_0 \left(\frac{n(n-1)}{2} - \mathbb{F}_0^{n-1} \right) \quad (87)$$

$$= \delta_0 + \eta_0 \mathbb{F}_{(n-1)(\frac{n}{2}-1)}^{(n-1)(\frac{n}{2})}, \quad (88)$$

and the assertion follows. \blacksquare

APPENDIX F PROOF OF THEOREM 6

Let's assume that for $n_T \neq 1, n-1$, an $\sum_{i=0}^{n_T} \binom{n}{i}$ -EBM exists on half-lines $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$. Defining $n_c := \sum_{i=0}^{n_T} \binom{n}{i}$ and analogous to Eq. (35) in Appendix D, $\forall \mathbb{X} \subseteq \mathbb{F}_1^n$,

$$\mathbb{S}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} a_i \mid \mathbb{Y} \subseteq \mathbb{X}, |\mathbb{Y}| \leq n_T, \mathbb{Y} \neq \emptyset \right\}, \quad (89)$$

$$\mathbb{T}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} \frac{|a_i|}{\eta} \mid \mathbb{Y} \subseteq \mathbb{X}, |\mathbb{Y}| \leq n_T, \mathbb{Y} \neq \emptyset \right\}, \quad (90)$$

$$\mathbb{D}(\mathbb{X}) := \left\{ \sum_{i \in \mathbb{Y}} c_i \mid \mathbb{Y} \subseteq \mathbb{X}, |\mathbb{Y}| \leq n_T, \mathbb{Y} \neq \emptyset \right\}, \quad (91)$$

where $\{c_i | i \in \mathbb{F}_1^n\}$ was defined in Eq. (49), we infer $\exists M \in \{0, 1 - n_c\}, \eta \in \mathbb{R}_{>0}$ such that $\eta \mathbb{F}_M^{M+n_c-1} = \{0\} \cup \mathbb{S}(\mathbb{F}_1^n)$.

Note that in Eqs. (89-91), \mathbb{X} can be any subset of \mathbb{F}_1^n , regardless of $|\mathbb{X}|$. Let's consider the proof procedure in Appendix D as an algorithm and replace $\mathbb{S}(\mathbb{X})$, $\mathbb{T}(\mathbb{X})$, $\mathbb{D}(\mathbb{X})$ and 2^n with $\mathbb{S}(\mathbb{X})$, $\mathbb{T}(\mathbb{X})$, $\mathbb{D}(\mathbb{X})$ and n_c , respectively, to make an equivalent algorithm (EA) for Appendix F. Denoting the corresponding corollaries in EA by prefix E , it can be seen that Corollary E -D.1 is based on Corollary E -D.1.1, while the proof of the latter requires $\sum_{i \in \mathbb{A}} a_i, \sum_{i \in \mathbb{B}} a_i \in \{0\} \cup \mathbb{S}(\mathbb{F}_1^n)$. For this to happen, we only need to show $|\mathbb{A}|, |\mathbb{B}| \leq n_T$, which can be simply deduced from Eqs. (43,44), considering $|\mathbb{U}|, |\mathbb{V}| \leq n_T$ in Corollary E -D.1.1 and the fact that $L = 0$ or n for half-lines $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$, respectively. Thus, Corollary E -D.1 holds. Similarly, Corollaries E -D.2.1 and E -D.2.2 hold here for every step $k \in \mathbb{F}_1^{n-1}$, i.e., $\forall k \in \mathbb{F}_1^{n-1}, \forall x \in \{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\} : x \geq c_{k+1}$ and $c_{k+1} \in \{\mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k)\}$. As a result, the equivalent

$$\mathbb{S}(\mathbb{F}_1^{k+1}) = \mathbb{S}(\mathbb{F}_1^{k+1} \setminus ex) \cup (a_{ex} + \mathbb{S}(\mathbb{F}_1^{k+1} \setminus ex)) \cup \{a_{ex}\} \quad (55)$$

$$= \mathbb{G} \cup (\eta(-1)^{m'_{k+1}2^k} + \mathbb{G}) \cup \{\eta(-1)^{m'_{k+1}2^k}\} \quad (56)$$

$$= (\eta \mathbb{F}_{M''}^{2^k + M'' - 1} \setminus 0) \cup (\eta(-1)^{m'_{k+1}2^k} + (\eta \mathbb{F}_{M''}^{2^k + M'' - 1} \setminus 0)) \cup \{\eta(-1)^{m'_{k+1}2^k}\} \quad (57)$$

$$= (\eta \mathbb{F}_{M''}^{2^k + M'' - 1} \setminus 0) \cup \left(\eta \mathbb{F}_{(-1)^{m'_{k+1}2^k + M''}}^{(1+(-1)^{m'_{k+1}2^k + M'' - 1}} \setminus \eta(-1)^{m'_{k+1}2^k} \right) \cup \{\eta(-1)^{m'_{k+1}2^k}\} \quad (58)$$

$$= (\eta \mathbb{F}_{M'}^{2^{k+1} + M' - 1} \setminus 0) \Rightarrow \{0\} \cup \mathbb{S}(\mathbb{F}_1^{k+1}) = \eta \mathbb{F}_{M'}^{2^{k+1} + M' - 1}, \quad (59)$$

$$\begin{aligned} \binom{n-1}{n_T-1} \sum_{i=1}^n \left\lfloor \frac{a_i n_T - \delta}{\eta n_T} \right\rfloor &= \sum_{i=-K}^{-K+n_c-1} i = \frac{n_c(n_c-1)}{2} - K n_c \\ &= \frac{1}{2} \binom{n}{n_T} \left(\binom{n}{n_T} - 1 \right) - K \binom{n}{n_T} \end{aligned} \quad (76)$$

$$\Rightarrow 2 \binom{n-1}{n_T-1} \sum_{i=1}^n \left\lfloor \frac{a_i n_T - \delta}{\eta n_T} \right\rfloor = \left(\binom{n-1}{n_T} + \binom{n-1}{n_T-1} \right) \times \left(\binom{n}{n_T} - 1 - 2K \right) \quad (77)$$

$$\Rightarrow \binom{n-1}{n_T-1} \left| \binom{n-1}{n_T} \left(\binom{n-1}{n_T} + \binom{n-1}{n_T-1} - 1 - 2K \right) \right| \quad (78)$$

$$\Rightarrow \binom{n-1}{n_T-1} \left| \binom{n-1}{n_T} \left(\binom{n-1}{n_T} - 1 - 2K \right) \right| \quad (79)$$

$$\Rightarrow \exists q \in \mathbb{Z} \ni \frac{(n-1)!}{(n_T-1)!(n-n_T)!} q = \frac{(n-1)!}{(n_T)!(n-n_T-1)!} \times \left(\binom{n-1}{n_T} - 1 - 2K \right) \quad (80)$$

$$\Leftrightarrow q n_T = (n-n_T) \left(\binom{n-1}{n_T} - 1 - 2K \right) \quad (81)$$

$$\Rightarrow q n_T \stackrel{2}{\equiv} (n-n_T) \left(\binom{n-1}{n_T} - 1 - 2K \right) \quad (82)$$

$$\Rightarrow \binom{n-1}{n_T} \stackrel{2}{\equiv} 1 \text{ (Eq. (60) implies } n \stackrel{2}{\equiv} 1, n_T \stackrel{2}{\equiv} 0). \quad (83)$$

of Eq. (51) also holds as follows

$$\begin{aligned} \forall k \in \mathbb{F}_1^{n-1} : c_{k+1} &= \min \left\{ \mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^k) \right\}_{\leq n_T} \\ &= \min \left\{ \mathbb{F}_1^{n_c-1} \setminus \mathbb{D}(\mathbb{F}_1^k) \right\}_{\leq n_T}. \end{aligned} \quad (92)$$

Now we run the EA and follow the strong induction steps in Corollary E-D.2. We obtain the following three relations: $\mathbb{D}(\mathbb{F}_1^n) = \mathbb{F}_1^{n_c-1}$, $\forall i \in \mathbb{F}_1^{n_T+1} : c_i = 2^{i-1}$, and $c_{n_T+2} = \min_{\leq n_T} \{ \mathbb{D}(\mathbb{F}_1^n) \setminus \mathbb{D}(\mathbb{F}_1^{n_T+1}) \} = \min_{\leq n_T} \{ \mathbb{F}_1^{n_c-1} \setminus \mathbb{D}(\mathbb{F}_1^{n_T+1}) \}$. On the other hand, as c_i s in Eq. (49) are sorted in ascending order, $\min_{\leq n_T} \{ \mathbb{D}(\mathbb{F}_1^{n_T+1}) \} = c_1 = 1$, $\max_{\leq n_T} \{ \mathbb{D}(\mathbb{F}_1^{n_T+1}) \} = \sum_{i=2}^{n_T+1} c_i = 2^{n_T+1} - 2$ and $|\mathbb{D}(\mathbb{F}_1^{n_T+1})| = |\mathbb{D}(\mathbb{F}_1^{n_T+1})|_S = 2^{n_T+1} - 2$. Thus,

$$\mathbb{D}(\mathbb{F}_1^{n_T+1}) = \mathbb{F}_1^{2^{n_T+1}-2}, \quad (93)$$

and in a similar way, one can simply prove that

$$\mathbb{D}(\mathbb{F}_1^{n_T}) = \mathbb{F}_1^{2^{n_T}-2}. \quad (94)$$

Note that Eq. (94) can also be deduced from Eq. (93) by replacing n_T with $n_T - 1$. So $c_{n_T+2} = \min_{\leq n_T} \{ \mathbb{F}_1^{n_c-1} \setminus \mathbb{F}_1^{2^{n_T+1}-2} \} = 2^{n_T+1} - 1$. Hereafter, Corollary E-D.2 stops at calculating c_{n_T+3} , as the induction assumption in the previous step is not fulfilled due to $c_{n_T+2} = 2^{n_T+1} - 1 \neq 2^{n_T+1}$. Also, it is easy to prove that

$$\forall k \in \mathbb{F}_1^{n-1} : \mathbb{D}(\mathbb{F}_1^{k+1}) = \mathbb{D}(\mathbb{F}_1^k) \bigcup_{\leq n_T} \left(c_{k+1} + \mathbb{D}(\mathbb{F}_1^k) \right) \bigcup_{\leq n_T-1} \{ c_{k+1} \} \quad (95)$$

by elimination and re-addition of element c_{k+1} —similar to Eq. (55)—and applying the constraint on the number of elements. Note that Eq. (95) has two independent parameters k and n_T . Thus, the following relation also holds

$$\forall k \in \mathbb{F}_1^{n-1} : \mathbb{D}(\mathbb{F}_1^{k+1}) = \mathbb{D}(\mathbb{F}_1^k) \bigcup_{\leq n_T-1} \left(c_{k+1} + \mathbb{D}(\mathbb{F}_1^k) \right) \bigcup_{\leq n_T-2} \{ c_{k+1} \}. \quad (96)$$

Substituting $k = n_T + 1$ and $k = n_T$ into Eqs. (95,96), respectively, yields

$$\mathbb{D}(\mathbb{F}_1^{n_T+2}) = \mathbb{D}(\mathbb{F}_1^{n_T+1}) \bigcup_{\leq n_T} \left(c_{n_T+2} + \mathbb{D}(\mathbb{F}_1^{n_T+1}) \right) \bigcup_{\leq n_T-1} \{ c_{n_T+2} \}, \quad (97)$$

$$\mathbb{D}(\mathbb{F}_1^{n_T+1}) = \mathbb{D}(\mathbb{F}_1^{n_T}) \bigcup_{\leq n_T-1} \left(c_{n_T+1} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \bigcup_{\leq n_T-2} \{ c_{n_T+1} \}. \quad (98)$$

Before proceeding with the procedure above, we prove Corollary F.1.

Corollary F.1. $3(2^{n_T}) - 2 \in \mathbb{F}_1^{n_c-1}$.

Proof F.1: We need to prove that $\forall n, n_T \ni 1 < n_T < n - 1 : 3(2^{n_T}) - 2 \leq n_c - 1 = \sum_{i=0}^{n_T} \binom{n}{i} - 1$. Considering the increasing property of $\sum_{i=0}^{n_T} \binom{n}{i}$, i.e., $\sum_{i=0}^{n_T} \binom{n+1}{i} = \sum_{i=0}^{n_T} \binom{n}{i} + \binom{n}{i-1}$, the minimum of $\sum_{i=0}^{n_T} \binom{n}{i}$ occurs at $n = n_T + 2$, according to $n_T < n - 1$. So it is sufficient to prove that $3(2^{n_T}) - 1 \leq \sum_{i=0}^{n_T} \binom{n_T+2}{i}$, where $\sum_{i=0}^{n_T} \binom{n_T+2}{i} = \sum_{i=0}^{n_T+2} \binom{n_T+2}{i} - \binom{n_T+2}{n_T+1} - \binom{n_T+2}{n_T+2} = 4(2^{n_T}) - n_T - 3$. So the inequality reduces to

$3(2^{n_T}) - 1 \leq 4(2^{n_T}) - n_T - 3 \Leftrightarrow n_T + 2 \leq 2^{n_T} \quad \forall n_T \geq 2$, that can be proved by induction. ■

By substituting Eq. (98) into Eq. (97) and using Eq. (93,94), Eqs. (99-104) follow.

Now, from Eq. (92) for $k = n_T + 2$ and Eq. (104), we obtain Eqs. (105,106).

Note that without Corollary F.1, $\mathbb{F}_1^{n_c-1}$ in Eq. (106) would be meaningless and wrong. From Eq. (106) and $n_T > 1$, we have

$$c_{n_T+3} + c_1 = 3(2^{n_T}) - 1 \in \mathbb{D}(\mathbb{F}_1^n). \quad (107)$$

Also, from Eq. (104), $3(2^{n_T}) - 1 \in \mathbb{D}(\mathbb{F}_1^{n_T+2})$, i.e.,

$$\exists \mathbb{Y}_o \subseteq \mathbb{F}_1^{n_T+2} \ni |\mathbb{Y}_o| \leq n_T, \sum_{i \in \mathbb{Y}_o} c_i = 3(2^{n_T}) - 1. \quad (108)$$

Equations (107,108) imply that $c_{n_T+3} + c_1 = \sum_{i \in \mathbb{Y}_o} c_i$, and since $|\mathbb{D}(\mathbb{F}_1^n)|_S = |\mathbb{D}(\mathbb{F}_1^n)|$, it must hold that $\{1, n_T + 3\} = \mathbb{Y}_o$, while according to Eq. (108), $\mathbb{Y}_o \subseteq \mathbb{F}_1^{n_T+2}$. The assertion follows by contradiction. ■

REFERENCES

- [1] P. Ghofrani and A. Schmeink, "A fast energy-aware multi-target detection technique using binary wireless sensors," in *Wireless Communication Systems (ISWCS), IEEE Int. Symp. on*, pp. 252–257, Aug. 2017.
- [2] R. Hoshyar, F. P. Wathan, and R. Tafazolli, "Novel low-density signature for synchronous cdma systems over awgn channel," *IEEE Trans. Signal Process.*, vol. 56, pp. 1616–1626, Apr. 2008.
- [3] X. Wang and R. Chen, "Adaptive bayesian multiuser detection for synchronous cdma with gaussian and impulsive noise," *IEEE Trans. Signal Process.*, vol. 48, pp. 2013–2028, Jul. 2000.
- [4] J. Uher, T. Wysocki, and B. Wysocki, "Review of distributed beamforming," *Journal of Telecommun. and Inform. Technology*, pp. 78–88, Jan. 2011.
- [5] Z. Ding, W. H. Chin, and K. K. Leung, "Distributed beamforming and power allocation for cooperative networks," *IEEE Trans. Wireless Commun.*, vol. 7, pp. 1817–1822, May 2008.
- [6] P. A. Hoeher and T. Wo, "Superposition modulation: myths and facts," *IEEE Commun. Mag.*, vol. 49, pp. 110–116, Dec. 2011.
- [7] T. Ho and D. Lun, *Network coding: an introduction*. Cambridge University Press, 2008.
- [8] X. Han and F. Gao, "Homomorphic error-control codes for linear network coding in packet networks," *China Communications*, vol. 14, pp. 178–189, Sept. 2017.
- [9] L. Guo, Z. Ning, Q. Song, F. Huang, and A. Jamalipour, "Joint encoding and grouping multiple node pairs for physical-layer network coding with low-complexity algorithm," *IEEE Trans. Veh. Technol.*, vol. 66, pp. 9275–9286, Oct. 2017.
- [10] X. Xu, C. Wang, Y. J. Zhu, X. Ma, and X. Zhang, "Block markov superposition transmission of short codes for indoor visible light communications," *IEEE Commun. Lett.*, vol. 19, pp. 359–362, Mar. 2015.
- [11] P. Prochazka, T. Uricar, D. Halls, and J. Sykora, "Relaying in butterfly networks: Superposition constellation design for wireless network coding," in *Communication Workshop (ICCW), IEEE Int. Conf. on*, pp. 2168–2174, Jun. 2015.
- [12] G. Song, X. Wang, and J. Cheng, "Signature design of sparsely spread code division multiple access based on superposed constellation distance analysis," *IEEE Access*, vol. PP, no. 99, pp. 1–1, 2017.
- [13] D. Jevtić, "Optimum multiple access coding," *SIAM Review*, vol. 33, no. 1, pp. 114–115, 1991.
- [14] W. F. Lunnon, "Integer sets with distinct subset-sums," *Mathematics of Computation*, vol. 50, no. 181, pp. 297–320, 1988.

- [15] R. K. Guy, "Sets of integers whose subsets have distinct sums," in *Theory and Practice of Combinatorics* (P. L. Hammer, A. Rosa, G. Sabidussi, and J. Turgeon, eds.), vol. 60 of *North-Holland Mathematics Studies*, pp. 141 – 154, North-Holland, 1982.
- [16] D. Jevtić, "Sum-distinct sets," *SIAM Review*, vol. 37, no. 1, pp. 99–100, 1995.
- [17] D. B. Jevtic, "Disjoint uniquely decodable codebooks for noiseless synchronized multiple-access adder channels generated by integer sets," *IEEE Transactions on Information Theory*, vol. 38, pp. 1142–1146, May 1992.
- [18] D. B. Jevtić, "On families of sets of integral vectors whose representatives form sum-distinct sets," *SIAM Journal on Discrete Mathematics*, vol. 8, Nov. 1995.
- [19] P. Ghofrani and A. Schmeink, "3-D energy optimal receiver placement with constraints on the los delay and angle," *IEEE Transactions on Wireless Communications*, vol. 18, pp. 2156–2169, Apr. 2019.
- [20] P. Ghofrani, T. Wang, and A. Schmeink, "A fast converging channel estimation algorithm for wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 66, pp. 3169–3184, Jun. 2018.
- [21] Y. Isukapalli, J. Zheng, and B. D. Rao, "Optimum codebook design and average sep loss analysis of spatially independent and correlated feedback based miso systems with rectangular qam constellation," *IEEE Trans. Signal Process.*, vol. 57, pp. 2017–2024, May 2009.
- [22] M. Gharavi-Alkhansari and A. B. Gershman, "On diversity and coding gains and optimal matrix constellations for space-time block codes," *IEEE Trans. Signal Process.*, vol. 53, pp. 3703–3717, Oct. 2005.
- [23] A. Canteaut, G. Effinger, S. Huczynska, D. Panario, and L. Storme, *Contemporary Developments in Finite Fields and Applications*. World Scientific, Jun. 2016.



Anke Schmeink Anke Schmeink (M11-SM18) received the Diploma degree in mathematics with a minor in medicine and the Ph.D. degree in electrical engineering and information technology from RWTH Aachen University, Germany, in 2002 and 2006, respectively. She worked as a research scientist for Philips Research before joining RWTH Aachen University in 2008 where she is a professor since 2012. She spent several research visits with the University of Melbourne, and with the University of York. Anke Schmeink is a member of the Young Academy at the North Rhine-Westphalia Academy of Science. Her research interests are in information theory, machine learning, data analytics and optimization with focus on wireless communications and medical applications.



Pouya Ghofrani Pouya Ghofrani received the M.Sc. degree in electrical engineering and communications from RWTH Aachen University, Aachen, Germany, in 2015. He joined the ISEK Research and Teaching Group of RWTH Aachen University in 2015, where he received his Ph.D. degree (with distinction) in electrical engineering and communications in 2020. His research interests include fast signal processing schemes in sensor networks, estimation and detection techniques, localization, artificial intelligence, and hardware implementation.

$$\mathbb{D}(\mathbb{F}_1^{n_T+2}) = \mathbb{D}(\mathbb{F}_1^{n_T+1}) \cup \left(c_{n_T+2} + \left(\mathbb{D}(\mathbb{F}_1^{n_T}) \cup \left(c_{n_T+1} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \cup \{c_{n_T+1}\} \right) \right) \cup \{c_{n_T+2}\} \quad (99)$$

$$= \mathbb{D}(\mathbb{F}_1^{n_T+1}) \cup \left(c_{n_T+2} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \cup \left(c_{n_T+2} + c_{n_T+1} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \cup \{c_{n_T+2} + c_{n_T+1}\} \cup \{c_{n_T+2}\} \quad (100)$$

$$= \mathbb{D}(\mathbb{F}_1^{n_T+1}) \cup \{c_{n_T+2}\} \cup \left(c_{n_T+2} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \cup \{c_{n_T+2} + c_{n_T+1}\} \cup \left(c_{n_T+2} + c_{n_T+1} + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \quad (101)$$

$$= \mathbb{F}_1^{2(2^{n_T})-2} \cup \{2(2^{n_T}) - 1\} \cup \left(2(2^{n_T}) - 1 + \mathbb{F}_1^{2^{n_T}-2} \right) \cup \{3(2^{n_T}) - 1\} \cup \left(3(2^{n_T}) - 1 + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \quad (102)$$

$$= \mathbb{F}_1^{2(2^{n_T})-1} \cup \mathbb{F}_{2(2^{n_T})}^{3(2^{n_T})-3} \cup \{3(2^{n_T}) - 1\} \cup \left(3(2^{n_T}) - 1 + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \quad (103)$$

$$= \mathbb{F}_1^{3(2^{n_T})-3} \cup \{3(2^{n_T}) - 1\} \cup \left(3(2^{n_T}) - 1 + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \quad (104)$$

$$c_{n_T+3} = \min \left\{ \mathbb{F}_1^{n_c-1} \setminus \mathbb{D}(\mathbb{F}_1^{n_T+2}) \right\} = \min \left\{ \left(\mathbb{F}_1^{n_c-1} \setminus \mathbb{F}_1^{3(2^{n_T})-3} \right) \setminus \left(\{3(2^{n_T}) - 1\} \cup \left(3(2^{n_T}) - 1 + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \right) \right\} \quad (105)$$

$$= \min \left\{ \mathbb{F}_{3(2^{n_T})-2}^{n_c-1} \setminus \underbrace{\left(\{3(2^{n_T}) - 1\} \cup \left(3(2^{n_T}) - 1 + \mathbb{D}(\mathbb{F}_1^{n_T}) \right) \right)}_{> 3(2^{n_T})-2} \right\} = 3(2^{n_T}) - 2. \quad (106)$$